# Synchronization by Nonlinear Frequency Pulling

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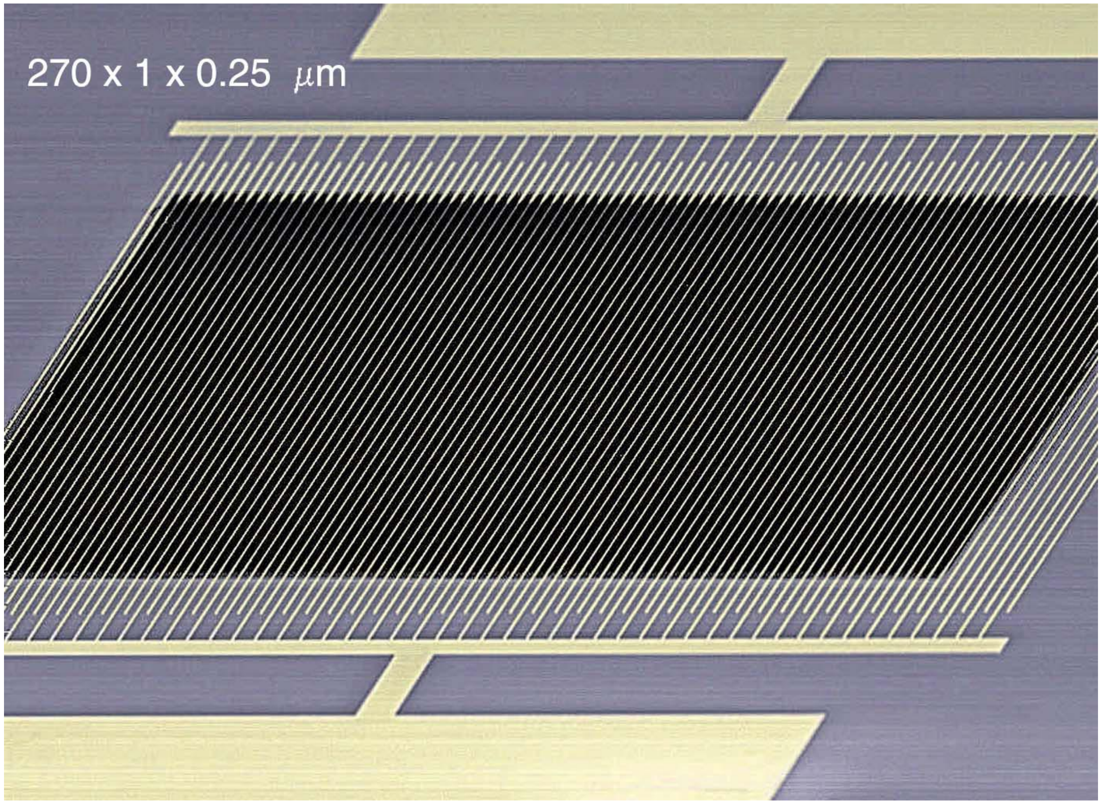
## Outline

- Motivation: MEMS and NEMS
- Phase and amplitude models of synchronization
- Model for reactive coupling and nonlinear frequency pulling
- Analysis and results
- Conclusions

Array of $\mu m$-scale oscillators [From Buks and Roukes (2002)]




Theory: R. Lifshitz R and MCC [Phys. Rev. B **67**, 134302 (2003 )]
*Response of parametrically driven nonlinear coupled oscillators with application to micromechanical and nanomechanical resonator arrays*

## MicroElectroMechanicalSystems and NEMS

Arrays of tiny mechanical oscillators:

- driven, dissipative $\Rightarrow$ nonequilibrium
- nonlinear
- collective
- noisy
- (potentially) quantum

Technological interest!

This talk: Synchronization

## Phase and amplitude models (mean field version)

- Phase model (Winfree-Kuramoto)

$$\dot{\theta}_n = \omega_n - K\frac{1}{N}\sum_m \sin(\theta_n - \theta_{n+m})$$

- Complex amplitude model $z_n = r_n e^{i\theta_n}$: dissipative coupling and saturating nonlinearity (Matthews, Mirollo, and Strogatz)

$$\dot{z}_n = i\omega_n z_n + (1 - |z_n|^2)z_n - K\frac{1}{N}\sum_m [z_n - z_m]$$

- Complex amplitude model: reactive coupling and nonlinear frequency pulling (see *Synchronization* by Pikovsky, Rosenblum, and Kurths)

$$\dot{z}_n = i(\omega_n - \alpha\,|z_n|^2)z_n + (1 - |z_n|^2)z_n - i\beta\frac{1}{N}\sum_m [z_n - z_m]$$

In each case $\omega_n$ is taken from some distribution $g(\omega)$ (eg. Lorentzian, Triangular, or top-hat) of width $w$.

## Connection with physical oscillator

$$0 = \ddot{x}_n + (1 + \omega_n)x_n - D[x_n - \tfrac{1}{2}(x_{n+1} + x_{n-1})] - \nu(1 - x_n^2)\dot{x}_n - a x_n^3$$

Assume dispersion, coupling, damping and nonlinear terms are small.

Introduce small parameter $\varepsilon$ and write

$$\omega_n = \varepsilon\bar{\omega}_n,\ D = \varepsilon\bar{D},\ a = \varepsilon\bar{a},\ \nu = \varepsilon\bar{\nu}$$

Then with the “slow” time scale $T = \varepsilon t$

$$x_n(t) = \left[A_n(T)e^{it} + c.c.\right] + \epsilon x_n^{(1)}(t) + \dots$$

This gives the complex amplitude model (nearest neighbor coupling) with

$$A_n \Rightarrow z_n, \qquad \bar{a} \Rightarrow \alpha, \qquad \bar{D} \Rightarrow \beta$$

## Synchronization

Order parameter

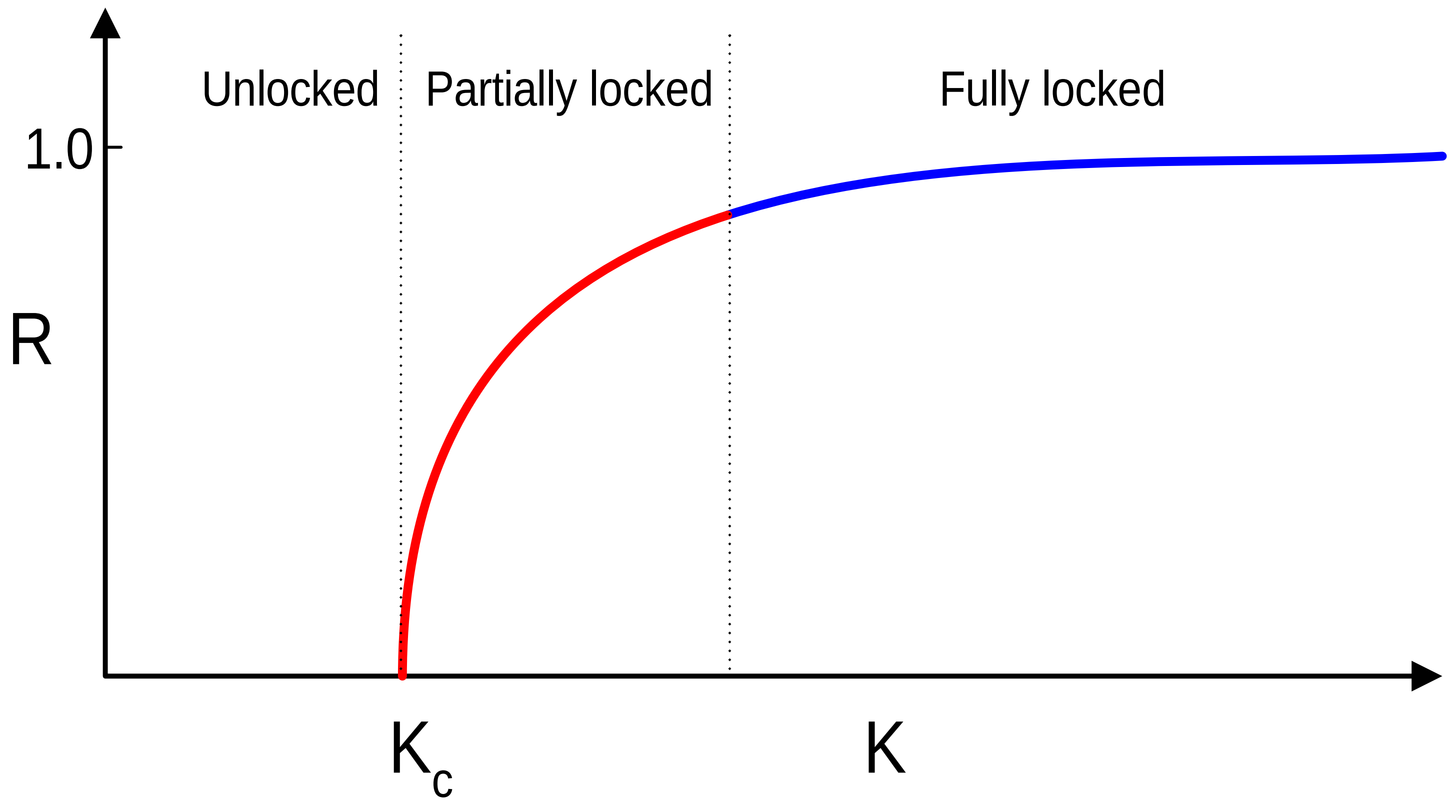
$$\Psi = N^{-1} \sum_n r_n \, e^{i\theta_n} = R \, e^{i\Theta}$$

(For phase model $r_n = 1$)

Synchronization occurs if $R \neq 0$.

- Fully locked state for all $\dot{\theta}_n = \dot{\Theta}$
- Partially locked state for some $\dot{\theta}_n = \dot{\Theta}$
- Novel state with $R \neq 0$ but no $\dot{\theta}_n = \dot{\Theta}$

## Results for the phase model (Kuramoto, 1975)



## Our model

$$\dot{z}_n = i(\omega_n - \alpha\,|z_n|^2)z_n + (1 - |z_n|^2)z_n + \frac{i\beta}{N}\sum_{m=1}^{N}(z_m - z_n)$$

Write as equations for magnitude and phase $z = re^{i\theta}$

$$\dot{\bar{\theta}}_n = \bar{\omega}_n + \alpha(1 - {r_n}^2) + \frac{\beta R}{r_n}\cos\bar{\theta}_n$$

$$\dot{r}_n = (1 - {r_n}^2)r_n + \beta R\sin\bar{\theta}_n$$

with $\bar{\theta}_n = \theta_n - \Theta$, $\bar{\omega}_n = \omega_n - \alpha - \beta - \dot{\Theta}$

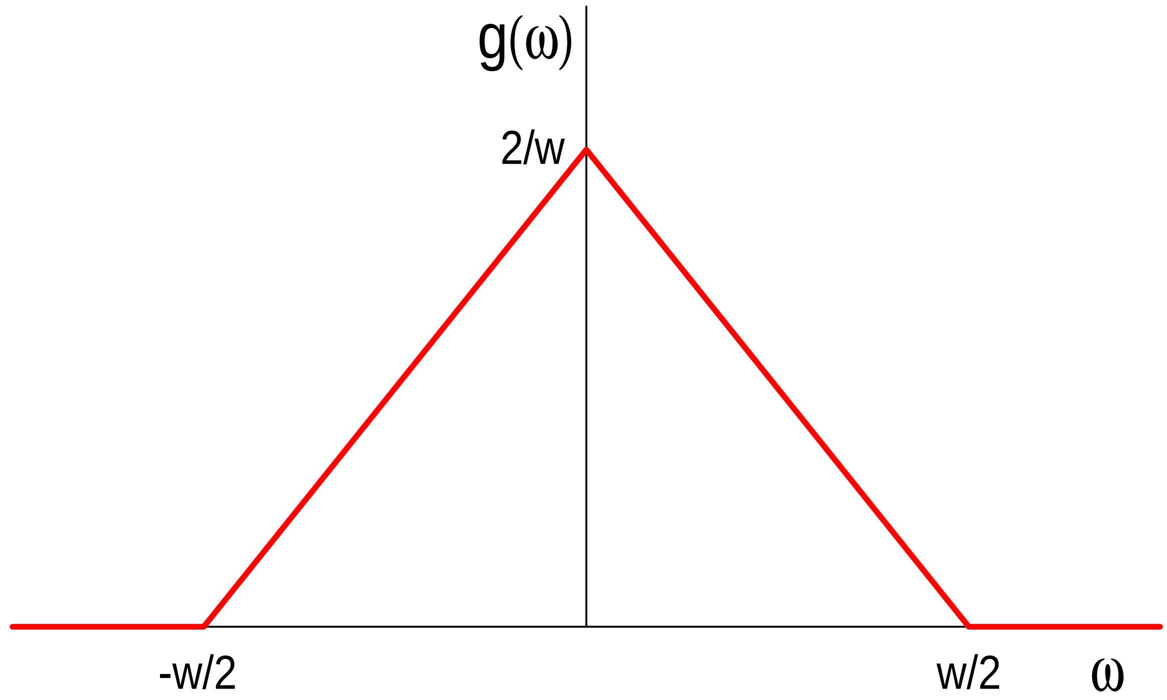
Self consistency condition

$$R = N^{-1}\sum_n r_n\, e^{i\bar{\theta}_n}$$

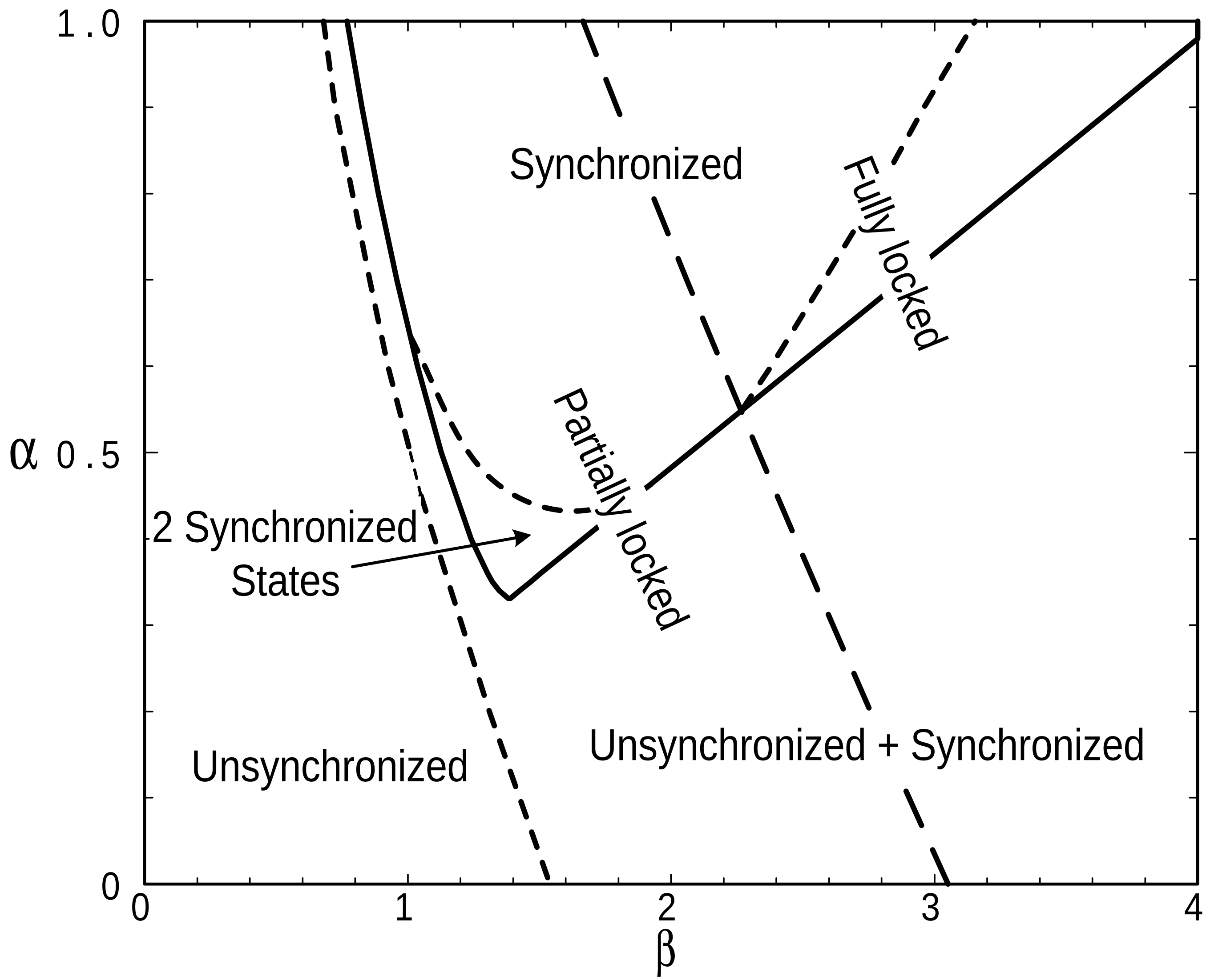
## Results

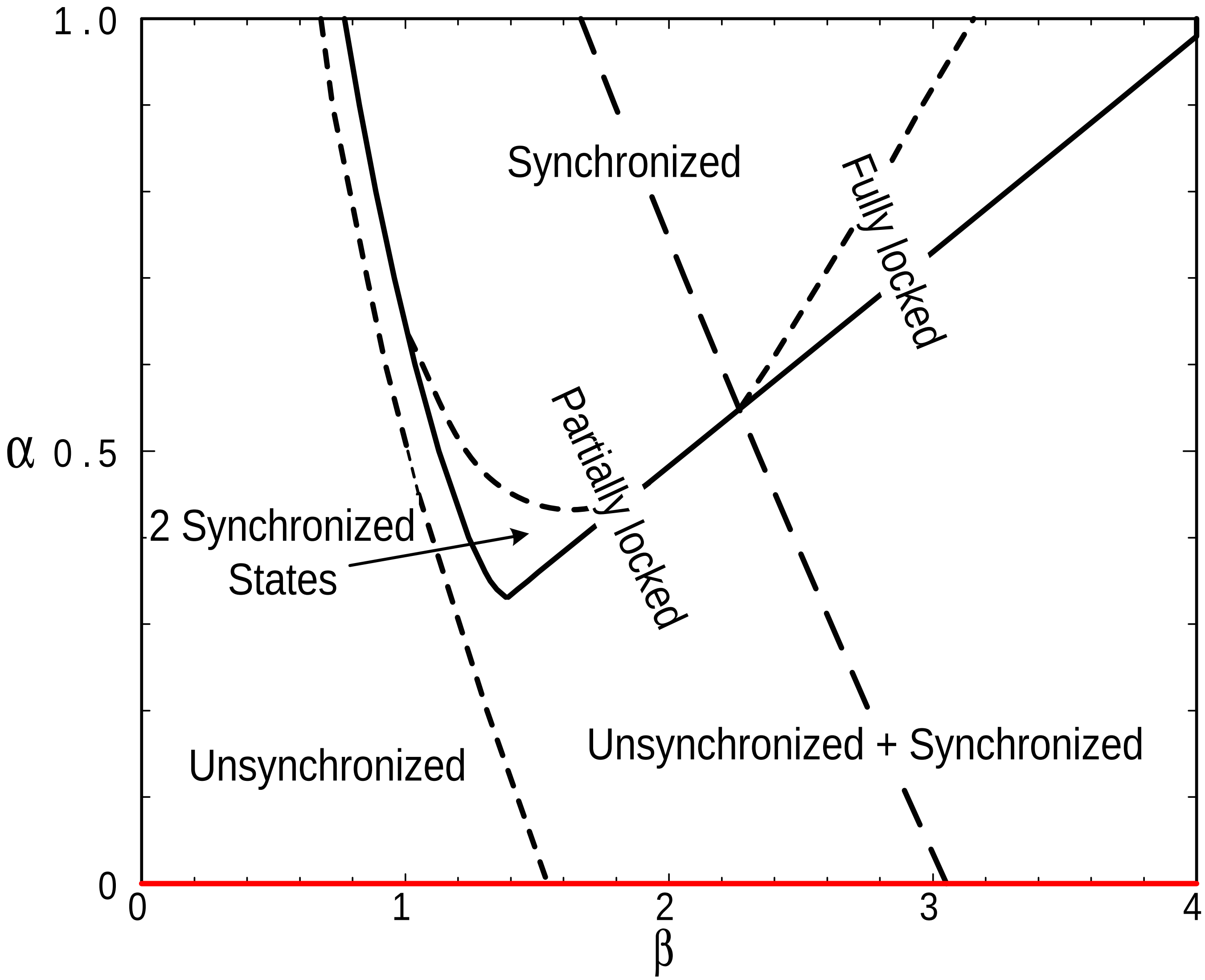
- Linear instability of unsynchronized $R = 0$ state (for Lorentzian, triangular, top-hat $g(\omega)$)
  - ◇ Order parameter frequency $\dot{\Theta}$ not trivially given by $g(\omega)$
  - ◇ For fixed $\alpha > \alpha_{\min}$ there are two values of $\beta$ giving linear instability
- Fully locked state
  - ◇ Again order parameter frequency $\dot{\Theta}$ not trivially given by $g(\omega)$
  - ◇ Linear instability may be through stationary or Hopf bifurcation
- Simulations of amplitude-phase model for up to 10000 oscillators with all-to-all coupling

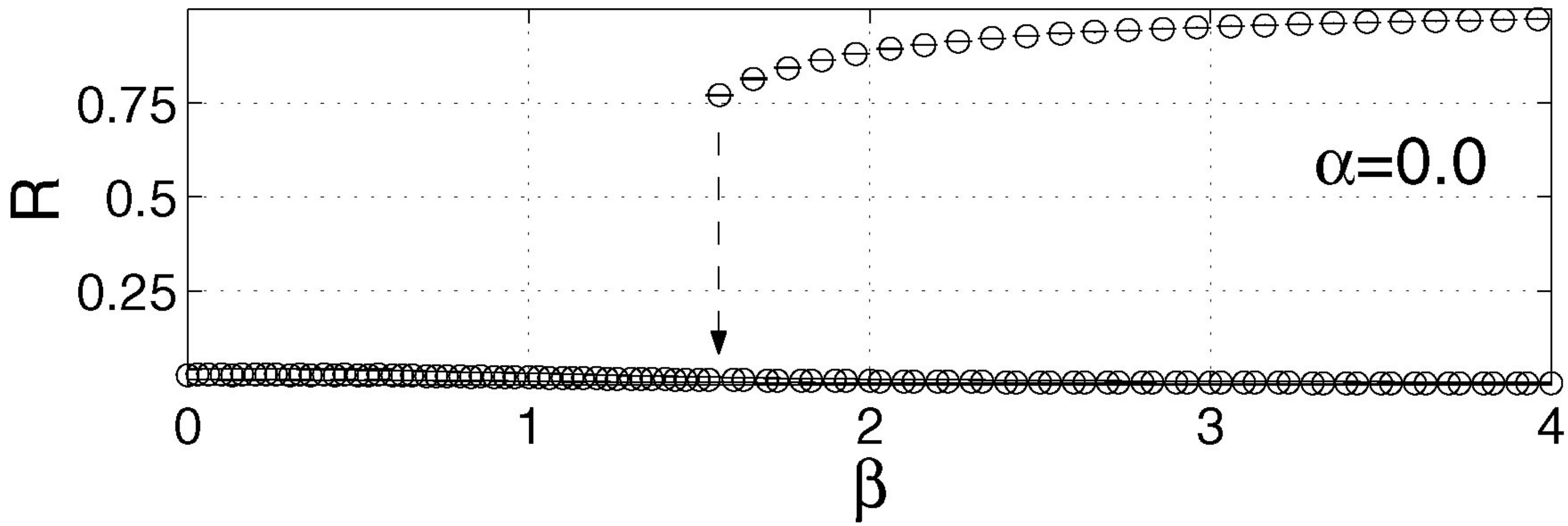
## Results for a triangular distribution

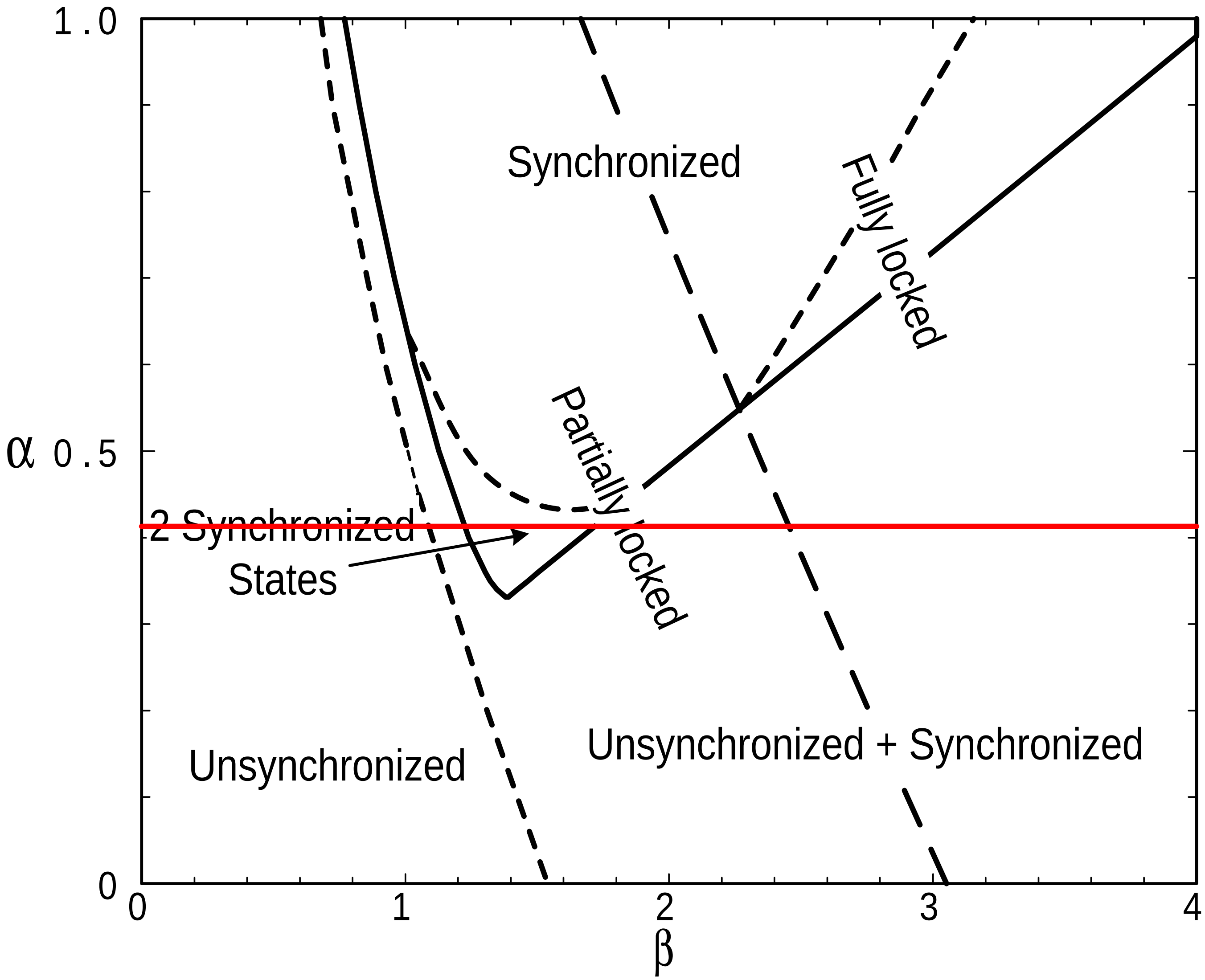


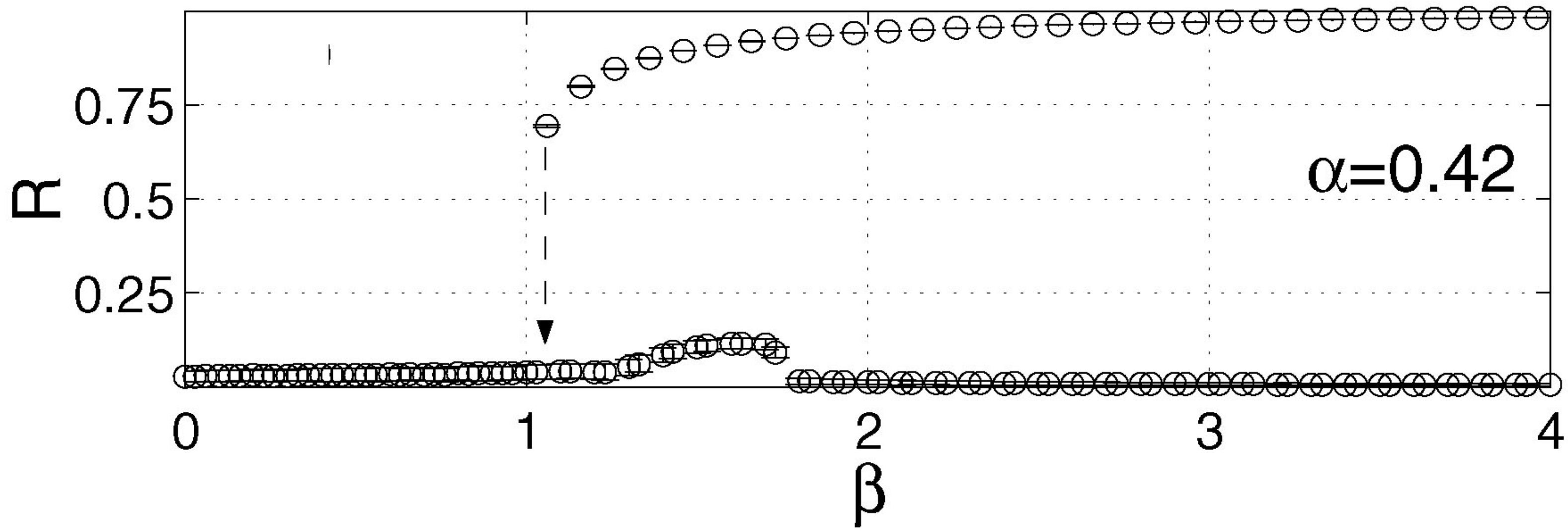

Show results for $w = 2\ldots$


1.0
α 0.5
0
0
1
2
3
4
β
Synchronized
Fully locked
Partially locked
2 Synchronized
States
Unsynchronized
Unsynchronized + Synchronized


1.0
α 0.5
0
0
1
2
3
4
β
Synchronized
Fully locked
Partially locked
2 Synchronized
States
Unsynchronized
Unsynchronized + Synchronized


0.75
0.5
0.25
R
0
1
2
3
4
β
α=0.0


1.0
α 0.5
0
0
1
2
3
4
β
Synchronized
Fully locked
Partially locked
2 Synchronized
States
Unsynchronized
Unsynchronized + Synchronized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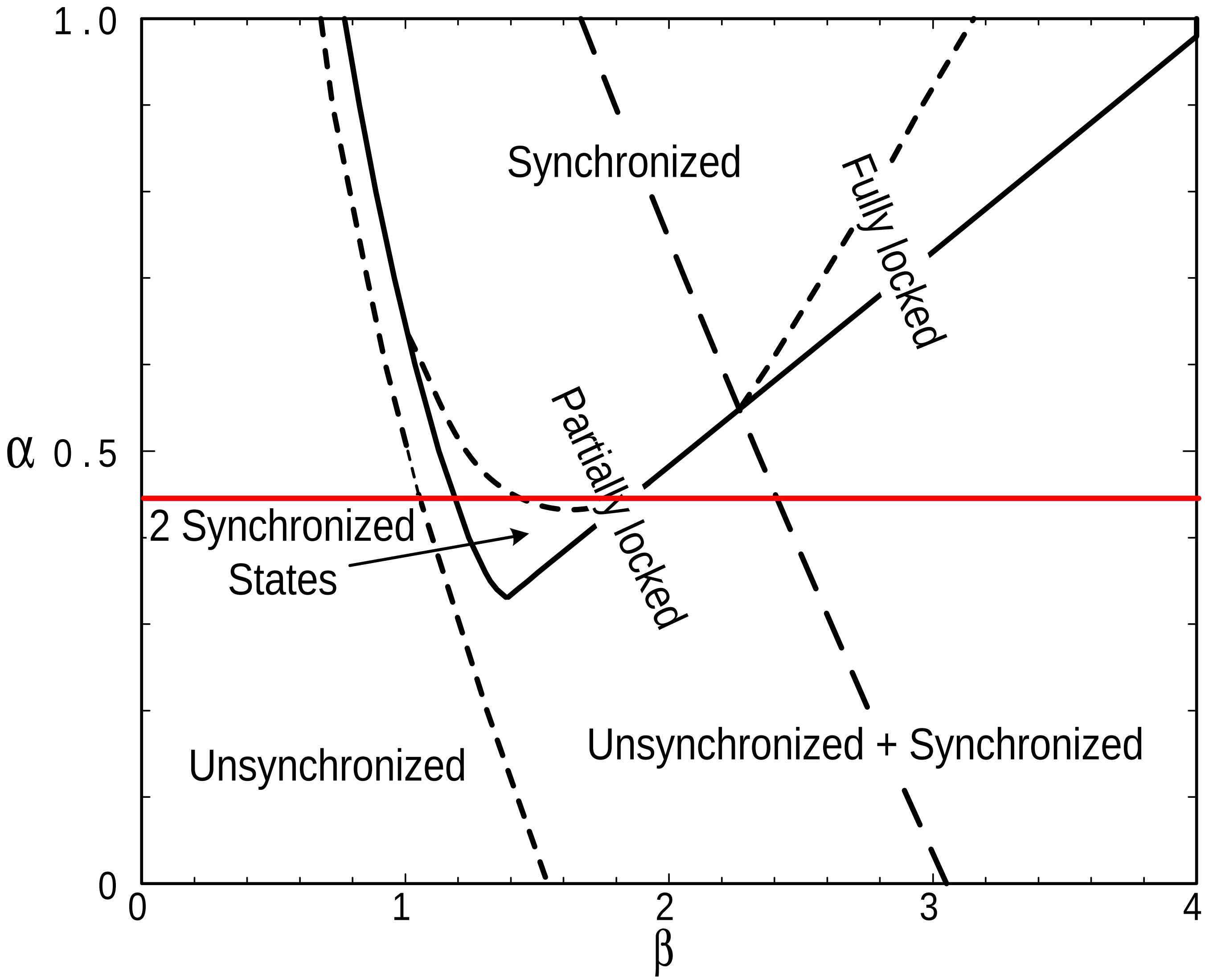
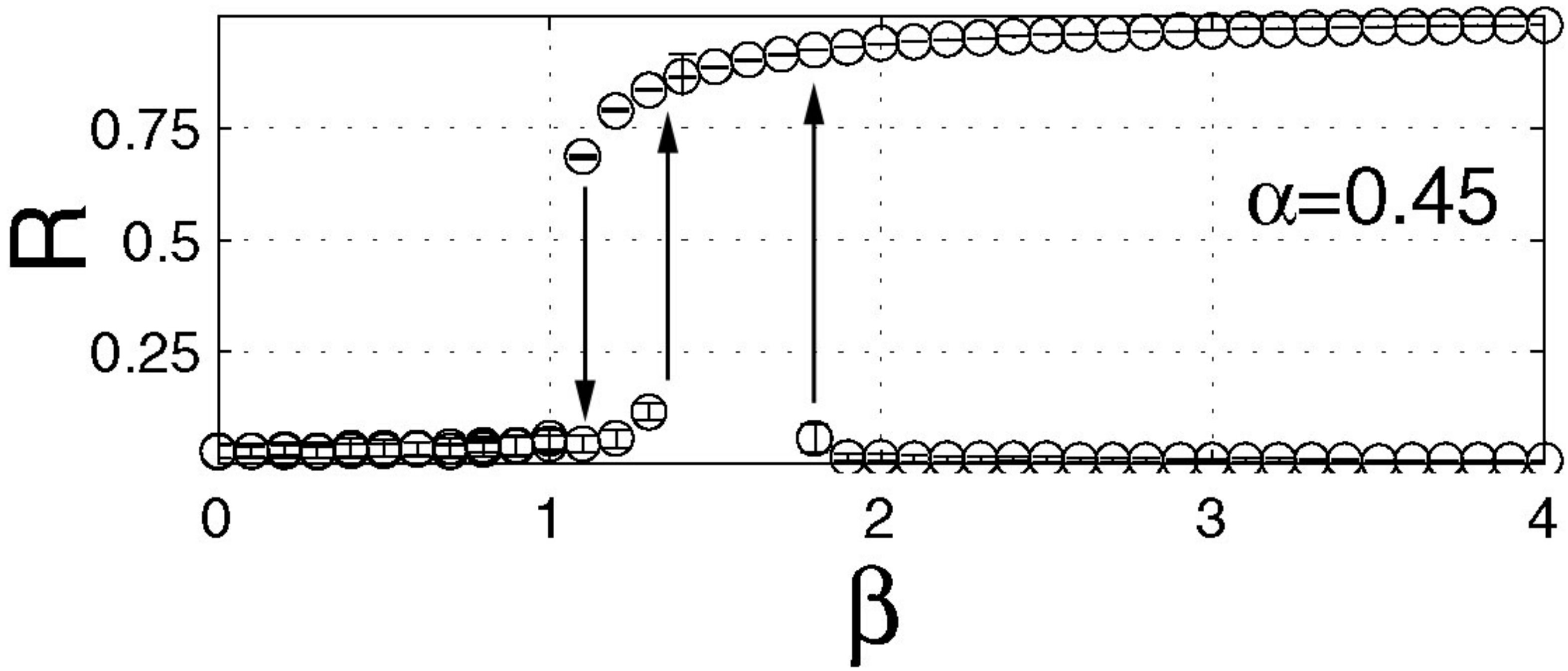
0.25
R
α=0.42
0
1
2
3
4
β


1.0
α 0.5
0
0
1
2
3
4
β
Synchronized
Fully locked
Partially locked
2 Synchronized
States
Unsynchronized
Unsynchronized + Synchronized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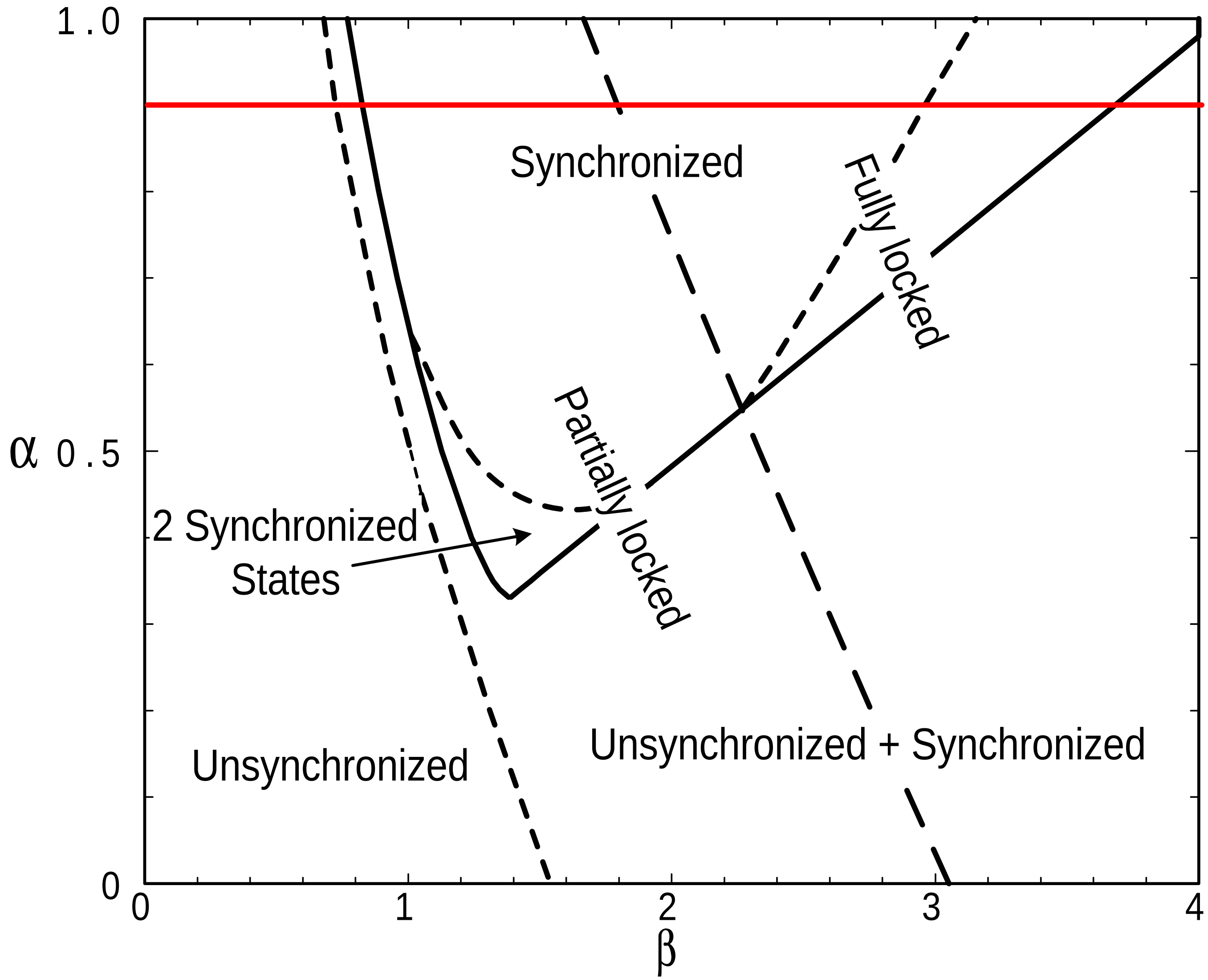
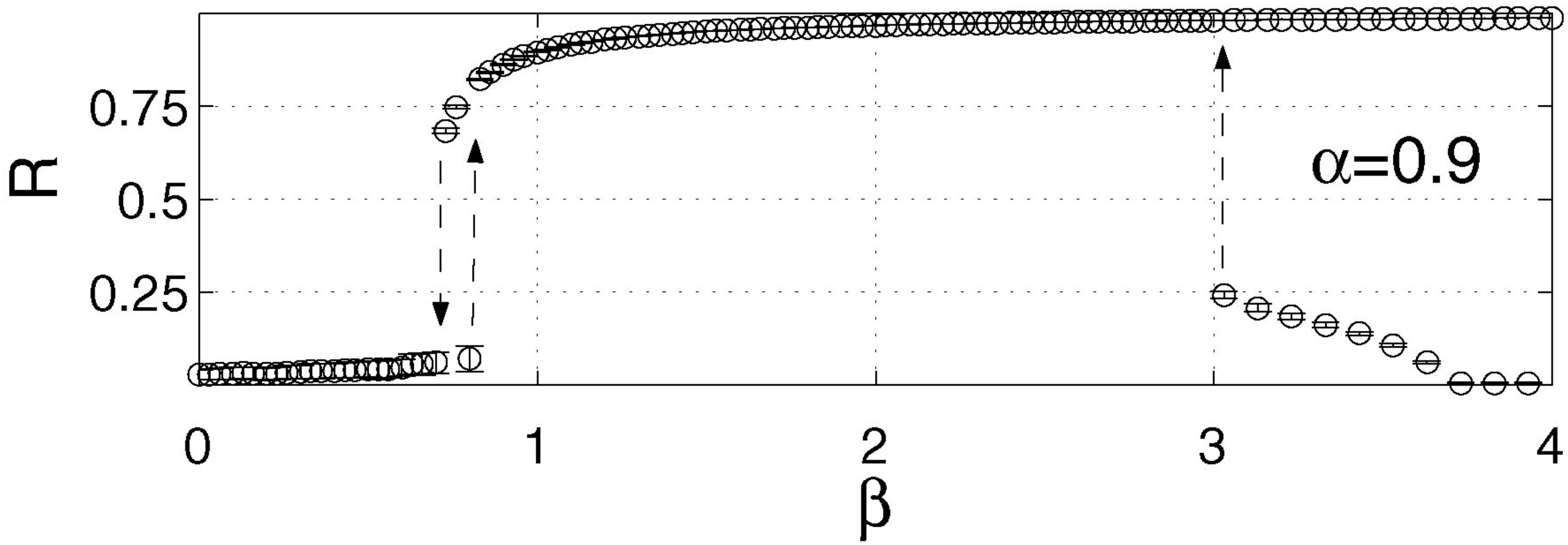
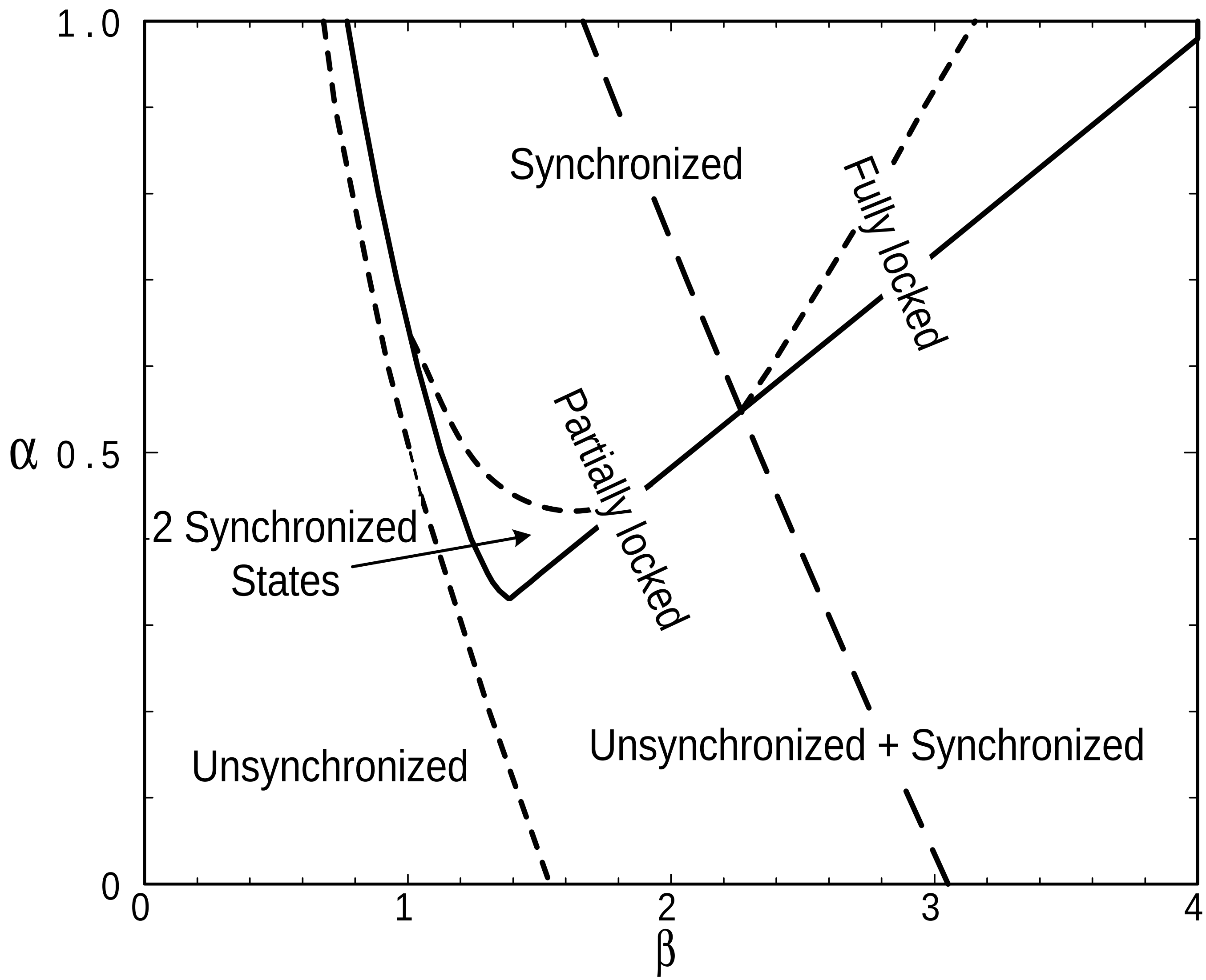
R
0.5
0.25
α=0.45
0
1
2
3
4
β


1.0
α 0.5
0
0
1
2
3
4
β
Synchronized
Fully locked
Partially locked
2 Synchronized
States
Unsynchronized
Unsynchronized + Synchronized


R
0.75
0.5
0.25
α=0.9
0
1
2
3
4
β


1.0
α 0.5
0
0
1
2
3
4
β
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Unsynchronized
Unsynchronized + Synchronized

# Conclusions

Micromechanical devices suggests a model for synchronization due to reactive coupling and nonlinear frequency pulling

Linear stability results for unsynchronized state (triangular, tophat, and Lorentzian distributions) and fully locked state (triangular, tophat)

Together with simulations yields a rich phase diagram of synchronized behavior

Novel state that is "synchronized" $R > 0$ but not frequency locked (no oscillator with $\dot{\theta} = \dot{\Theta}$)