Nonlinear and Collective Effects in Mesoscopic Mechanical Oscillators

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Support: NSF, Nato and EU, BSF, HRL

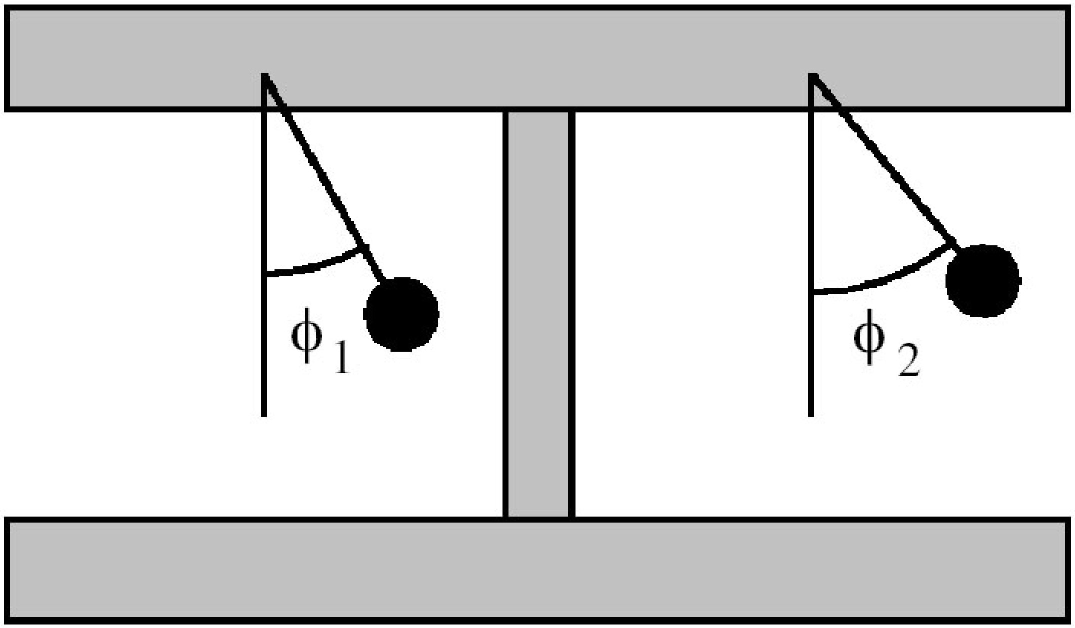
Message

- Arrays of micro- and nano-scale oscillators provide an interesting dynamical and pattern forming system
- Synchronization is an important feature of nonlinear oscillators and may be important technologically in MEMS/NEMS
- Synchronization in MEMS/NEMS motivates a model of synchronization involving nonlinear frequency pulling and reactive coupling

Outline

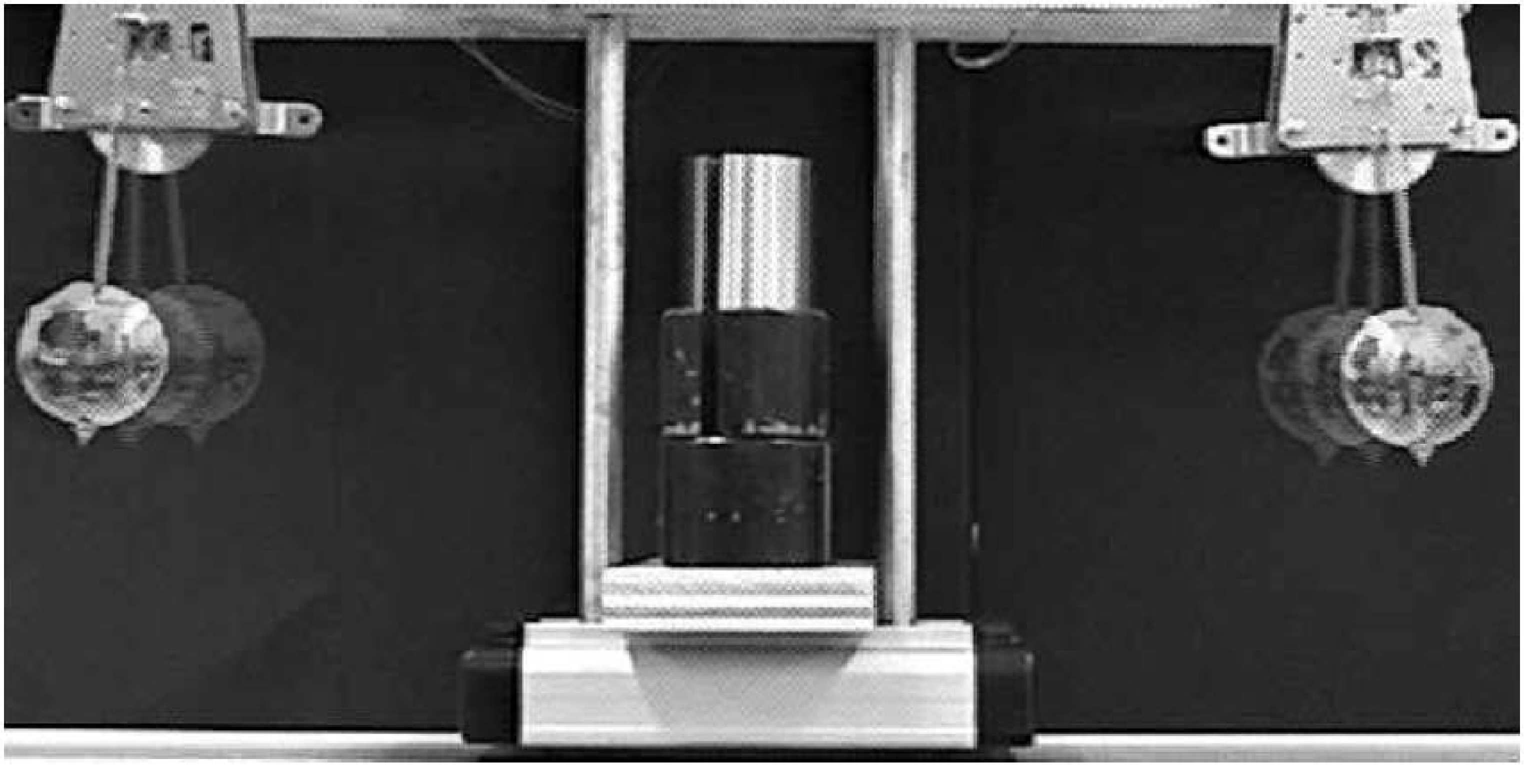
- Motivation: Huyghen's clocks and MEMS
- Phase and amplitude models
- Our model
- Analysis and results
  - ◇ onset of synchronization
  - ◇ full locking
  - ◇ simulations

Huyghen's Clocks (1665)



From: Matthew Bennett, Michael Schatz, Heidi Rockwood, and Kurt Wiesenfeld (Proc. Rot. Soc. Lond. 2002)

Experimental Reconstruction



Huyghen’s clocks:

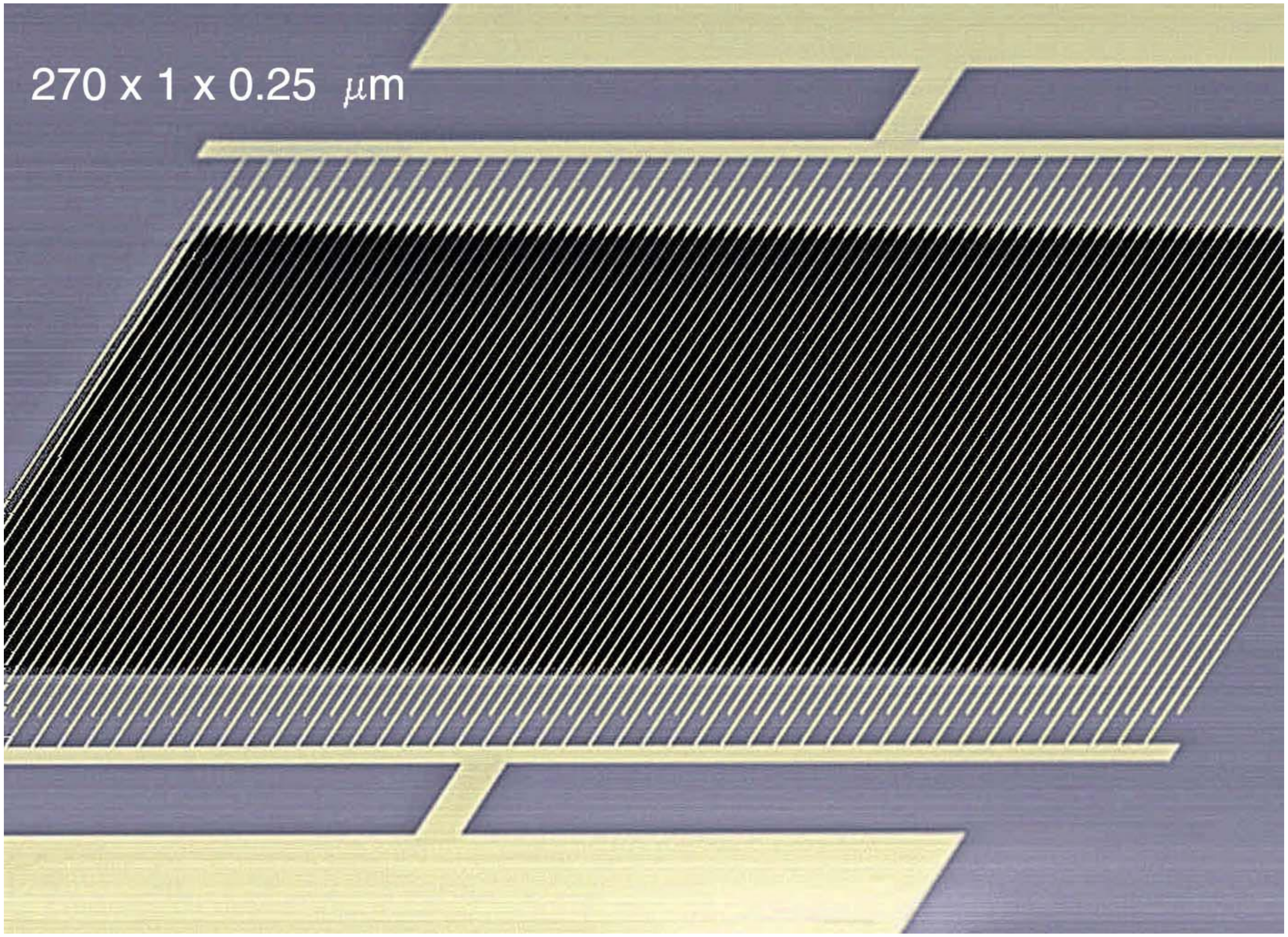
Model I: Synchronization occurs through increased dissipation acting on the in-phase motion

Alternative mechanism:

Model II: Synchronization occurs by nonlinear frequency tuning

cf. *Synchronization* by Arkady Pikovsky, Michael Rosenblum, Jürgen Kurths

Array of μm-scale oscillators



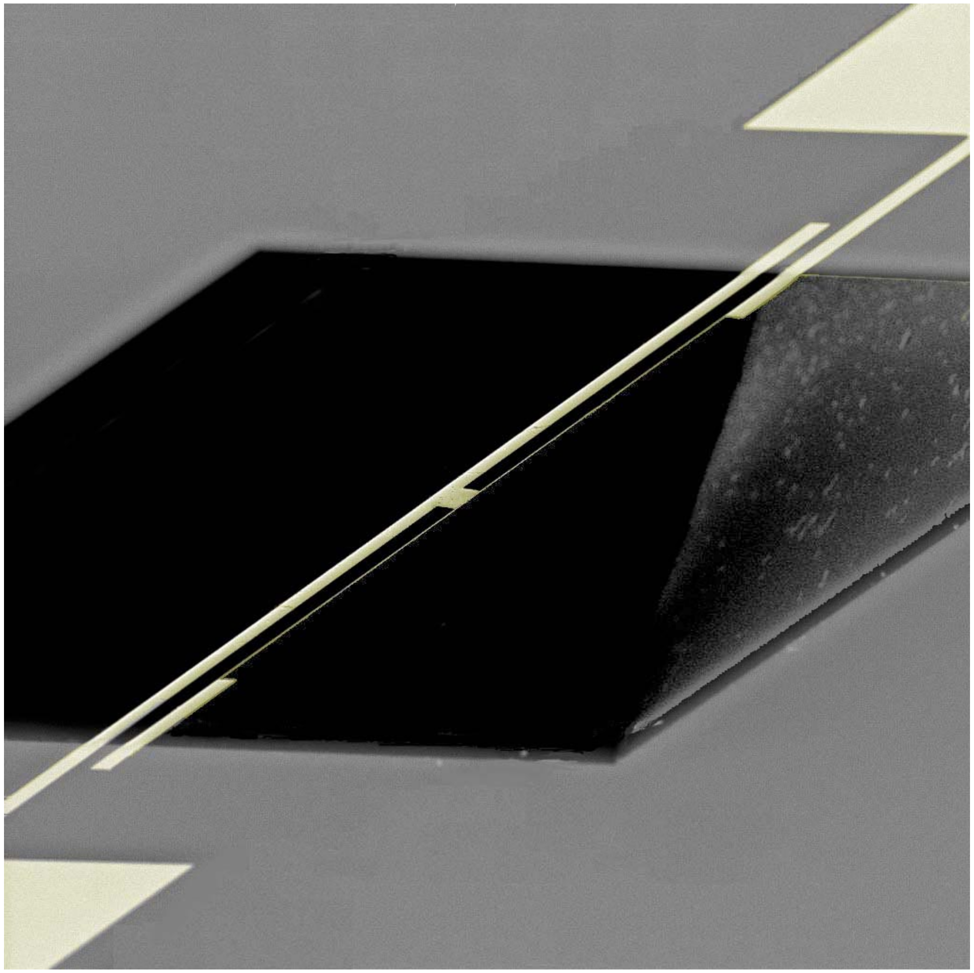
[From Buks and Roukes (2002)]

MicroElectroMechanicalSystems and NEMS

Arrays of tiny mechanical oscillators:

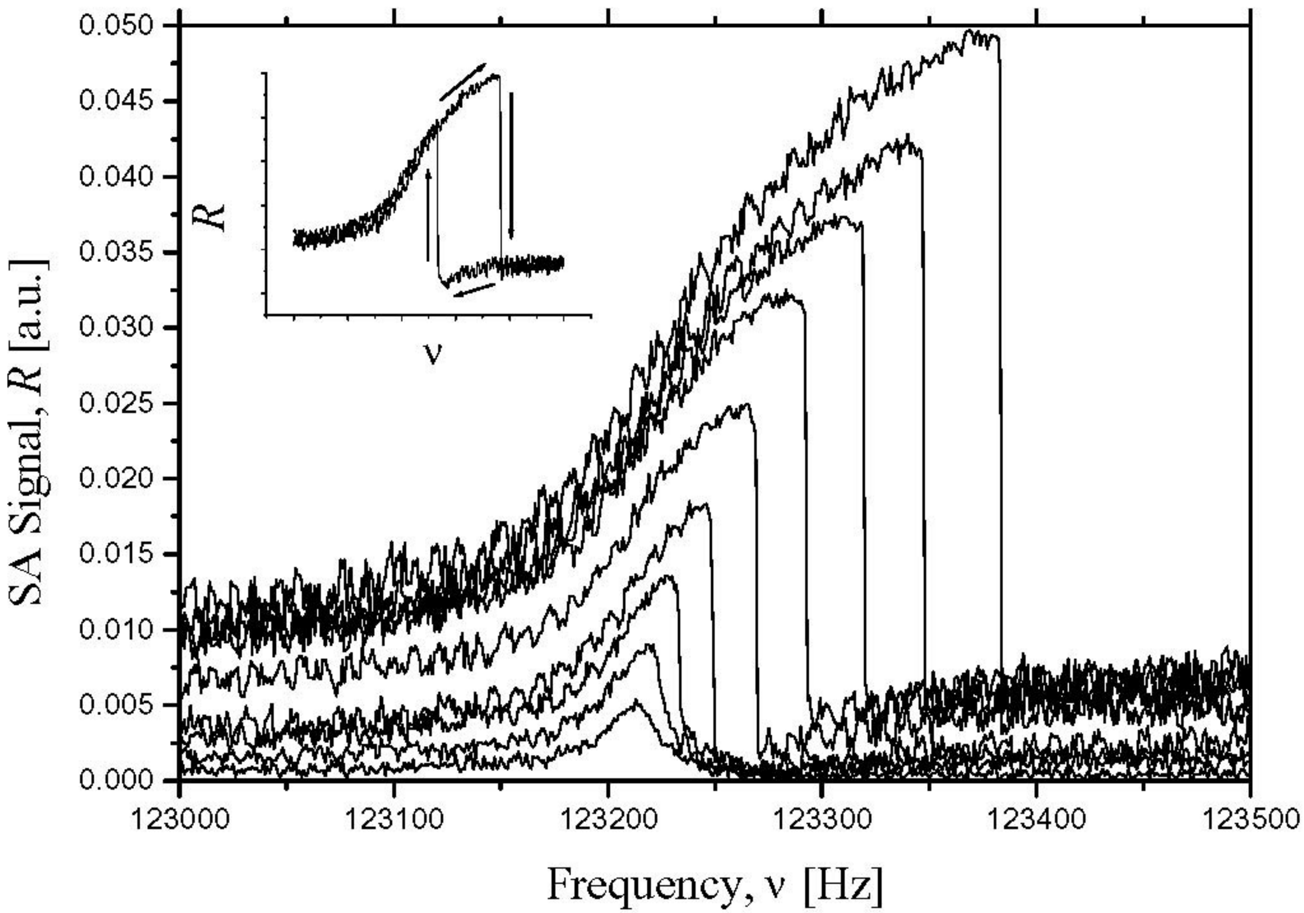
- driven, dissipative $\Rightarrow$ nonequilibrium
- nonlinear
- collective
- noisy
- (potentially) quantum

Technological interest!

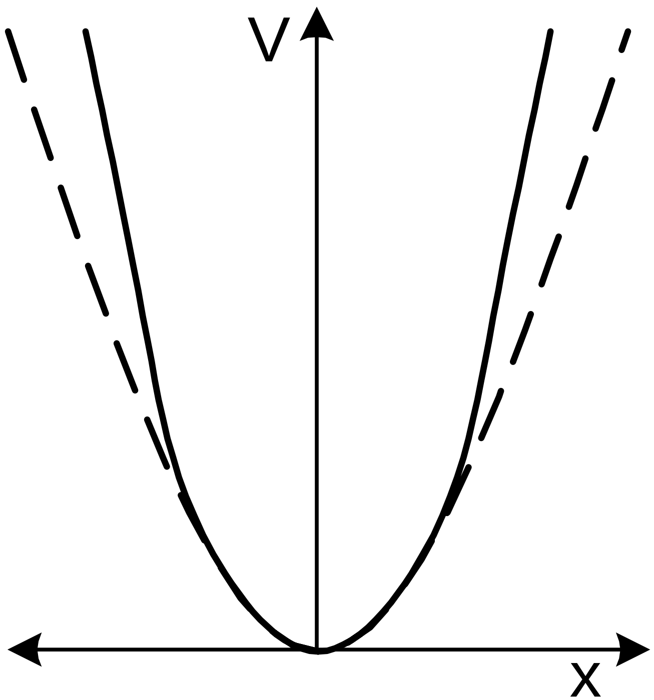


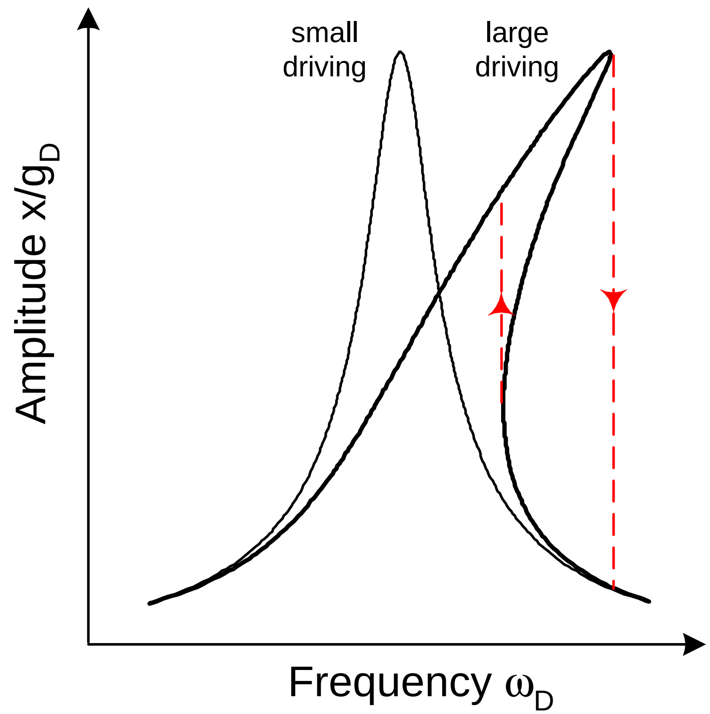
[From Buks and Roukes (2001)]

Oscillators are nonlinear... One oscillator



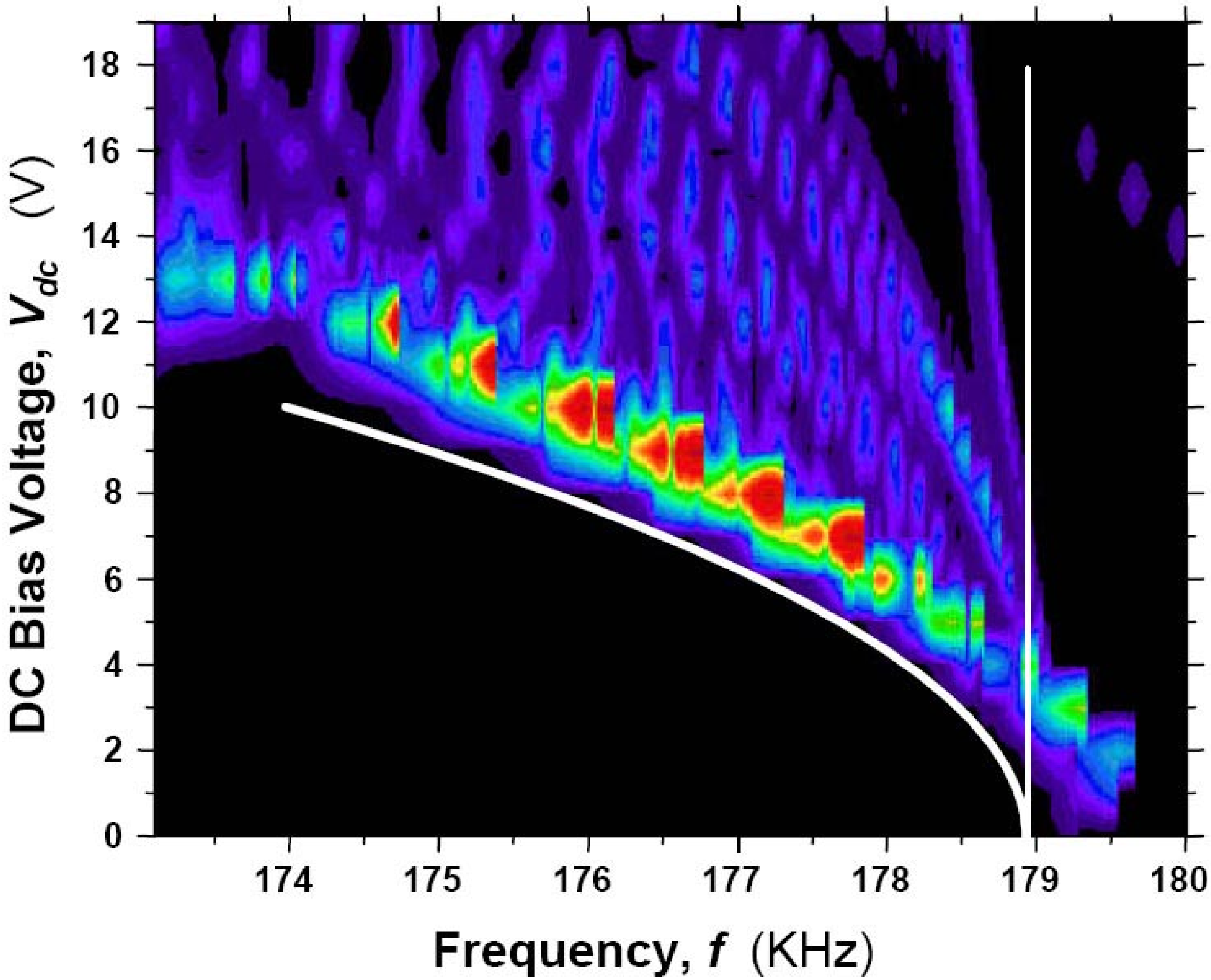
Duffing Oscillator





$$\ddot{x} + x + x^3 = g_D \cos \omega_D t$$

Sweeps of parametric drive frequency



Interesting behavior

- Suggestion of collective behavior
- Response beyond maximum linear frequency

R. Lifshitz and MCC [Phys. Rev. B **67**, 134302 (2003)]
Response of parametrically driven nonlinear coupled oscillators with application to micromechanical and nanomechanical resonator arrays

Model for synchronization

$$0 = \ddot{x}_n + (1 + \omega_n)x_n - D[x_n - \tfrac{1}{2}(x_{n+1} + x_{n-1})]$$

Model for synchronization

$$0 = \ddot{x}_n + (1 + \omega_n)x_n - D[x_n - \tfrac{1}{2}(x_{n+1} + x_{n-1})] - \nu(1 - x_n^2)\dot{x}_n - ax_n^3$$

Model for synchronization

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Assume dispersion, coupling, damping and nonlinear terms are small.

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Assume dispersion, coupling, damping and nonlinear terms are small.

Introduce small parameter ε and write

$$\omega_n = \varepsilon\bar{\omega}_n,\ D = \varepsilon\bar{D},\ a = \varepsilon\bar{a},\ \nu = \varepsilon\bar{\nu}$$

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Introduce small parameter ε and write

$$\omega_n = \varepsilon\bar{\omega}_n,\ D = \varepsilon\bar{D},\ a = \varepsilon\bar{a},\ \nu = \varepsilon\bar{\nu}$$

Then

$$x_n(t) = \left[z_n(T)e^{it} + c.c.\right] + \epsilon x_n^{(1)}(t) + \dots$$

with the "slow" time scale $T = \varepsilon t$.

Magnitude-phase description

$$\dot{z}_n = i(\omega_n - \alpha\,|z_n|^2)z_n + (1 - |z_n|^2)z_n + i\beta[z_n - \tfrac{1}{2}(z_{n+1} + z_{n-1})]$$

Magnitude-phase description

$$\dot{z}_n = i(\omega_n - \alpha\,|z_n|^2)z_n + (1 - |z_n|^2)z_n + i\beta[z_n - \tfrac{1}{2}(z_{n+1} + z_{n-1})]$$

Matthews, Mirollo, and Strogatz (1991) studied a similar model but with dissipative coupling $i\beta \to K$, and only saturating nonlinearity $\alpha = 0$

$$\dot{z}_n = i\omega_n z_n + (1 - |z_n|^2)z_n + K[z_n - \tfrac{1}{2}(z_{n+1} + z_{n-1})]$$

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$$\dot{z}_n = i\omega_n z_n + (1-|z_n|^2)z_n + K[z_n - \tfrac{1}{2}(z_{n+1}+z_{n-1})]$$

Phase model (Winfree, Kuramoto): $z_n \to 1 \times e^{i\theta_n}$

$$\dot{\theta}_n = \omega_n - \frac{K}{2}\sum_{m=\pm 1}\sin(\theta_n - \theta_{n+m})$$

Mean field model

$$\dot{\theta}_n = \omega_n - \frac{K}{N}\sum_{m}\sin(\theta_n - \theta_{n+m})$$

with ω_n taken from distribution $g(\omega)$.

Results for the phase model (Kuramoto, 1975)

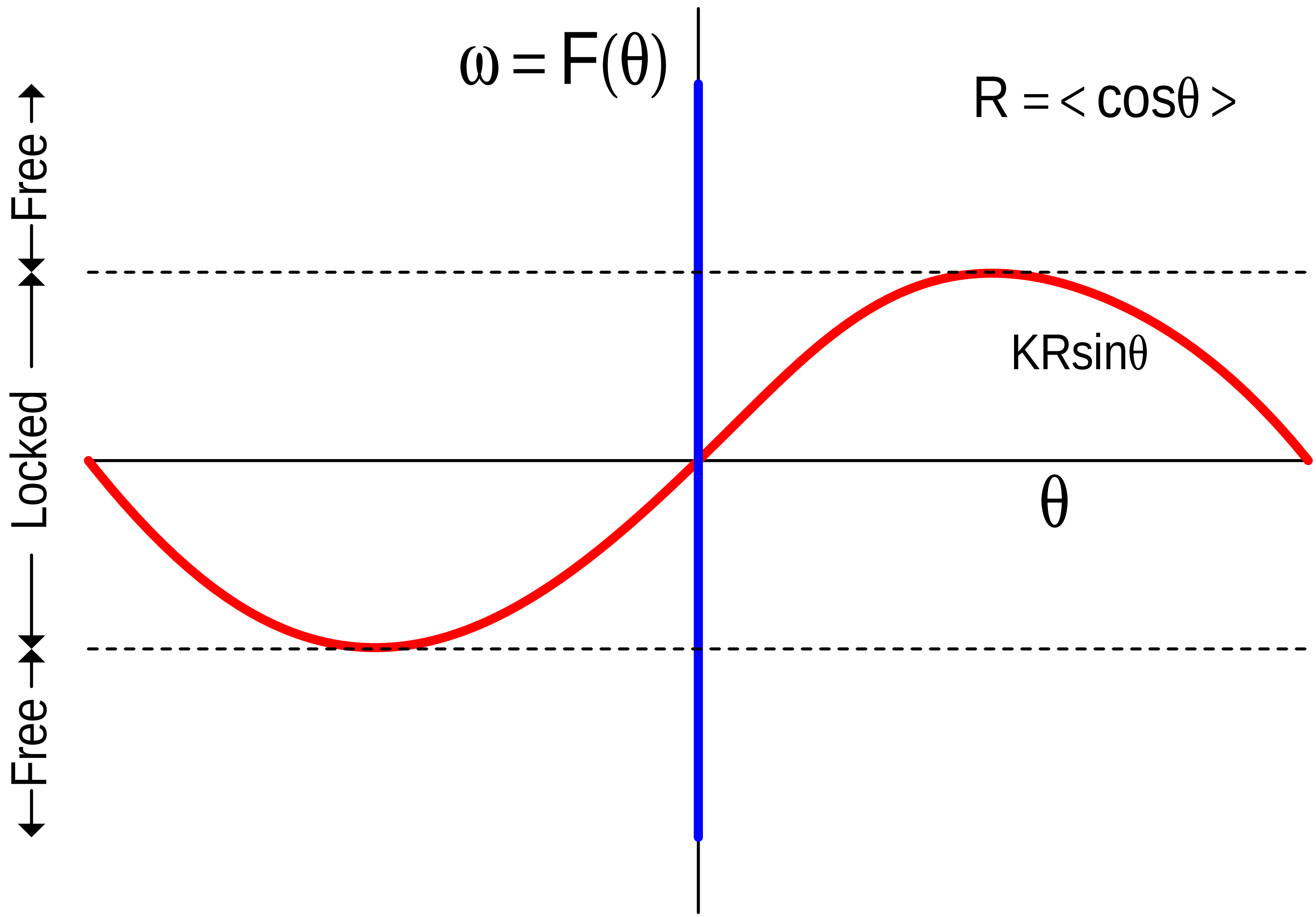
- define order parameter

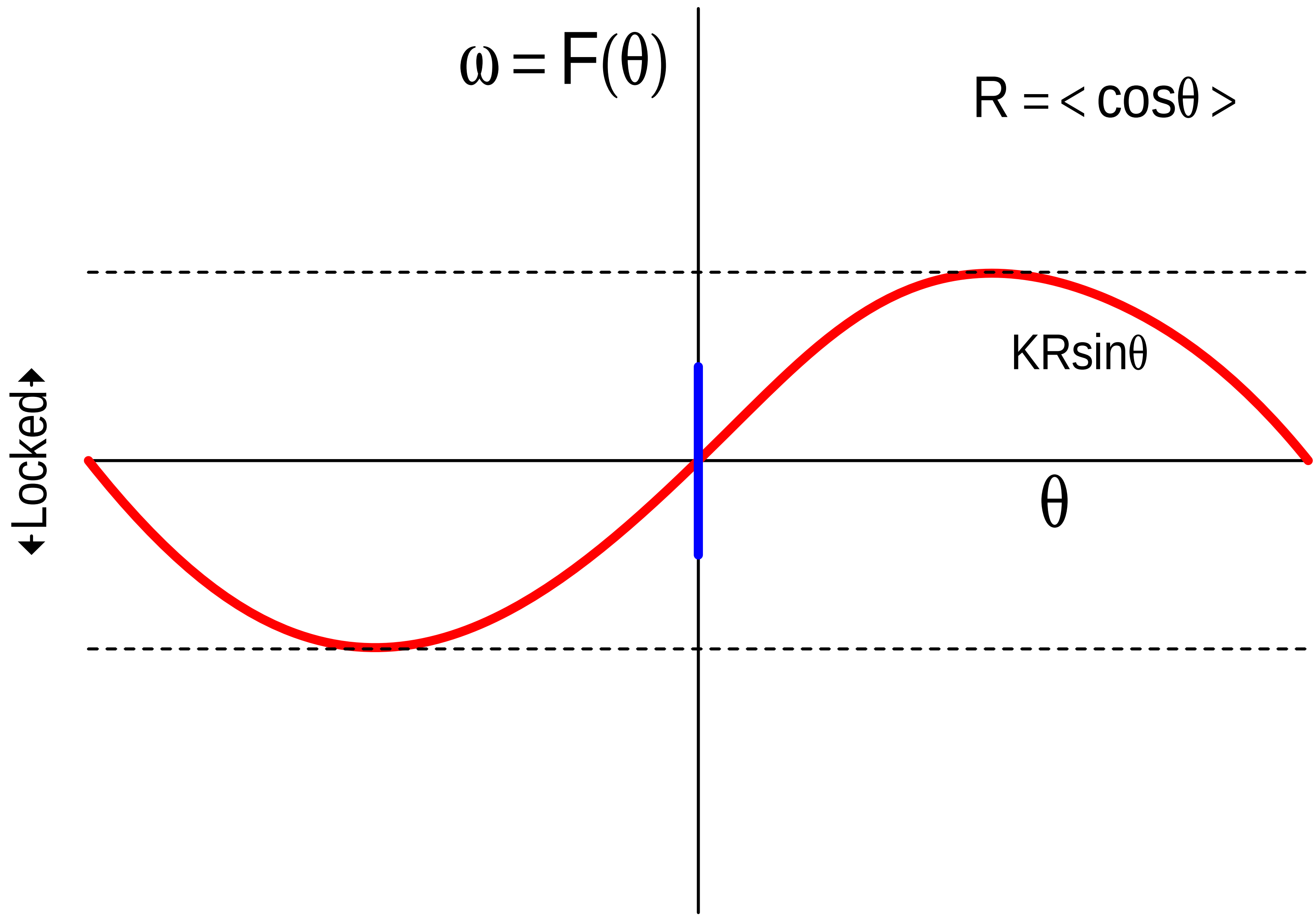
$$\Psi = N^{-1} \sum_n r_n \, e^{i\theta_n} = R \, e^{i\Theta}, \qquad r_n = 1$$

- synchronization occurs if $R \neq 0$
- phase evolution equation becomes

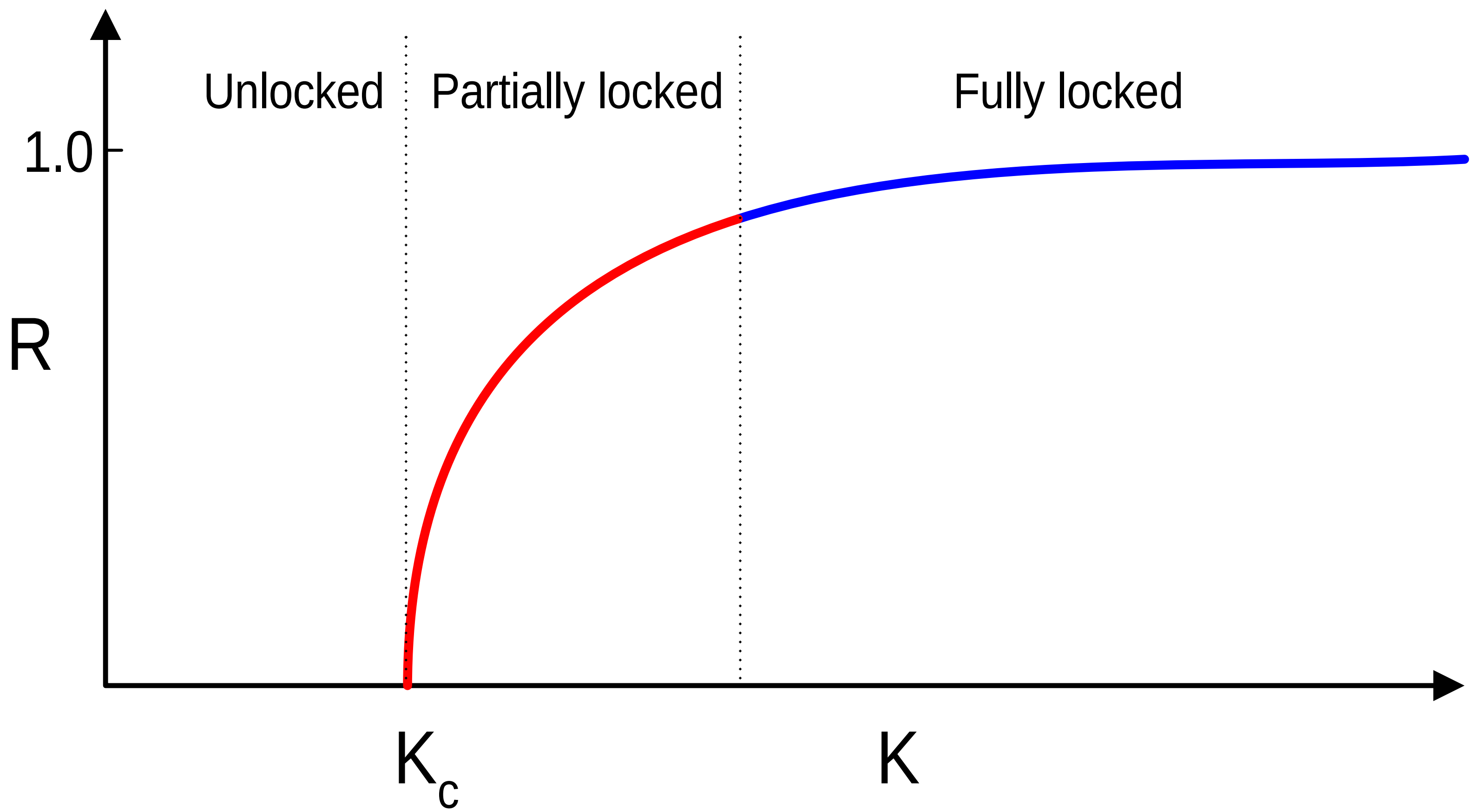
$$\dot{\theta}_n = \omega_n - F(\theta_n - \Theta) \qquad \text{with} \qquad F(\theta) = K \, R \sin\theta$$

- graphical solution…


$\omega = F(\theta)$
$R = \langle \cos\theta \rangle$
Free
Locked
Free
KRsinθ
θ



ω = F(θ)
R = < cosθ >
KRsinθ
Locked
θ


Unlocked
Partially locked
Fully locked
1.0
R
K_c
K

Our model (mean field version)

$$\dot{z}_n = i(\omega_n - \alpha\,|z_n|^2)z_n + (1-|z_n|^2)z_n + \frac{i\beta}{N}\sum_{m=1}^{N}(z_m - z_n)$$

Write as equations for magnitude and phase $z = re^{i\theta}$

$$\dot{\bar{\theta}} = \bar{\omega} + \alpha(1-r^2) + \frac{\beta R}{r}\cos\bar{\theta}$$
$$\dot{r} = (1-r^2)r + \beta R\sin\bar{\theta}$$

with $\bar{\theta} = \theta - \Theta$, $\bar{\omega} = \omega - \alpha - \beta - \dot{\Theta}$

Easy limit

The expectation of synchronization follows from the behavior for narrow frequency distributions and large α.

Magnitude relaxes rapidly to the value given by setting $\dot{r} = 0$

$$(1 - r^2)r = -\beta R \sin \bar{\theta}$$

If $r \simeq 1$ (OK for large α)

$$1 - r^2 \simeq -\beta R \sin \bar{\theta}$$

Phase equation becomes (ignoring βR compared with $\alpha\beta R$) Kuramoto equation with $K = \alpha\beta$

$$\dot{\bar{\theta}} \simeq \bar{\omega} - \alpha\beta R \sin \bar{\theta}$$

General results

$$\dot{\bar{\theta}} = \bar{\omega} + \alpha(1 - r^2) + \frac{\beta R}{r}\cos\bar{\theta}$$
$$\dot{r} = (1 - r^2)r + \beta R \sin\bar{\theta}$$

For locked oscillators

$$\dot{\bar{\theta}} = \dot{r} = 0$$

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For locked oscillators

$$\dot{\bar{\theta}} = \dot{r} = 0$$

- solve cubic equation for $r(\bar{\theta})$

General results

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For locked oscillators

$$\dot{\bar{\theta}} = \dot{r} = 0$$

- solve cubic equation for $r(\bar{\theta})$
- solve phase equation

$$\bar{\omega} = \frac{\beta R}{r(\bar{\theta})}(\alpha \sin \bar{\theta} - \cos \bar{\theta}) = F(\bar{\theta})$$

General results

$$\dot{\bar{\theta}} = \bar{\omega} + \alpha(1 - r^2) + \frac{\beta R}{r}\cos\bar{\theta}$$
$$\dot{r} = (1 - r^2)r + \beta R \sin\bar{\theta}$$

For locked oscillators

$$\dot{\bar{\theta}} = \dot{r} = 0$$

- solve cubic equation for $r(\bar{\theta})$
- solve phase equation

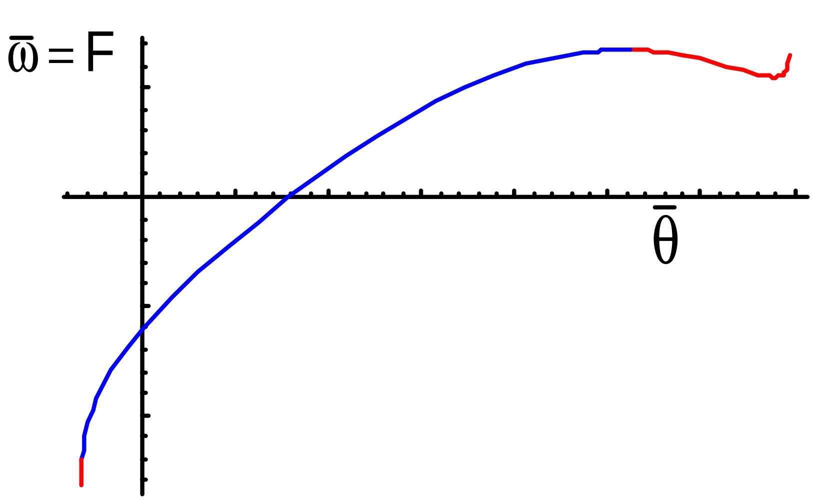
$$\bar{\omega} = \frac{\beta R}{r(\bar{\theta})}(\alpha \sin\bar{\theta} - \cos\bar{\theta}) = F(\bar{\theta})$$

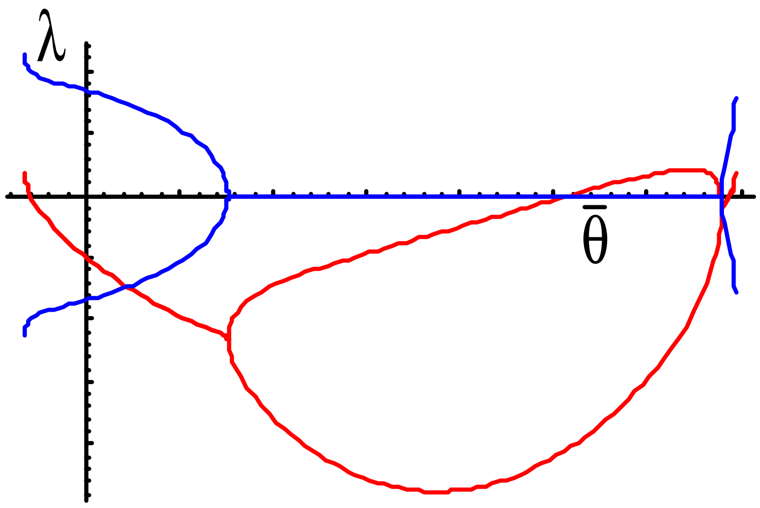
- test stability of solution $(r(\bar{\omega}), \bar{\theta}(\bar{\omega}))$

Results of analysis (α, $\beta > 0$)

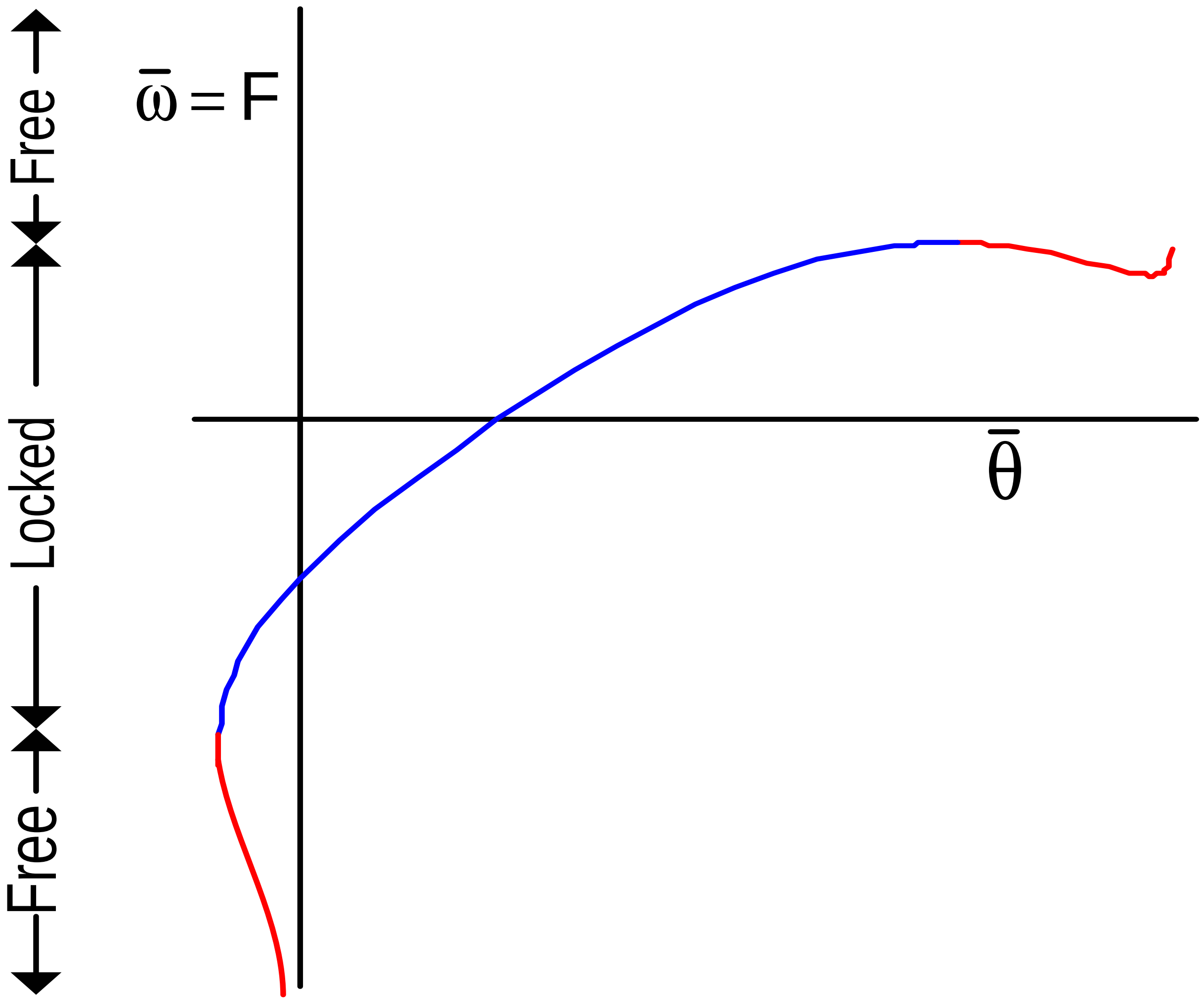
- There is always a stable solution at $\bar{\theta} = 0$, $r = 1$
- For $\beta R < 2/\sqrt{27}$ a real, positive solution for r exists for all $\bar{\theta}$ (but the solution is not necessarily stable).
- For $\beta R > 2/\sqrt{27}$ a real positive solution for r only exists for $-\bar{\theta}_B < \bar{\theta} < \pi + \bar{\theta}_B$ with $\theta_B = \sin^{-1}(2/\beta R\sqrt{27})$.
- Solutions with $r < 1/\sqrt{2}$ or $F'(\bar{\theta}) < 0$ are unstable
- Moving away from $\bar{\theta} = 0$ instability may occur through a stationary bifurcation ($\lambda = 0$) where $F'(\bar{\theta}) = 0$ or by a Hopf bifurcation ($\mathrm{Re}\,\lambda = 0$, $\mathrm{Im}\,\lambda \neq 0$) when $r = 1/\sqrt{2}$.

α=1.0, B=1.2

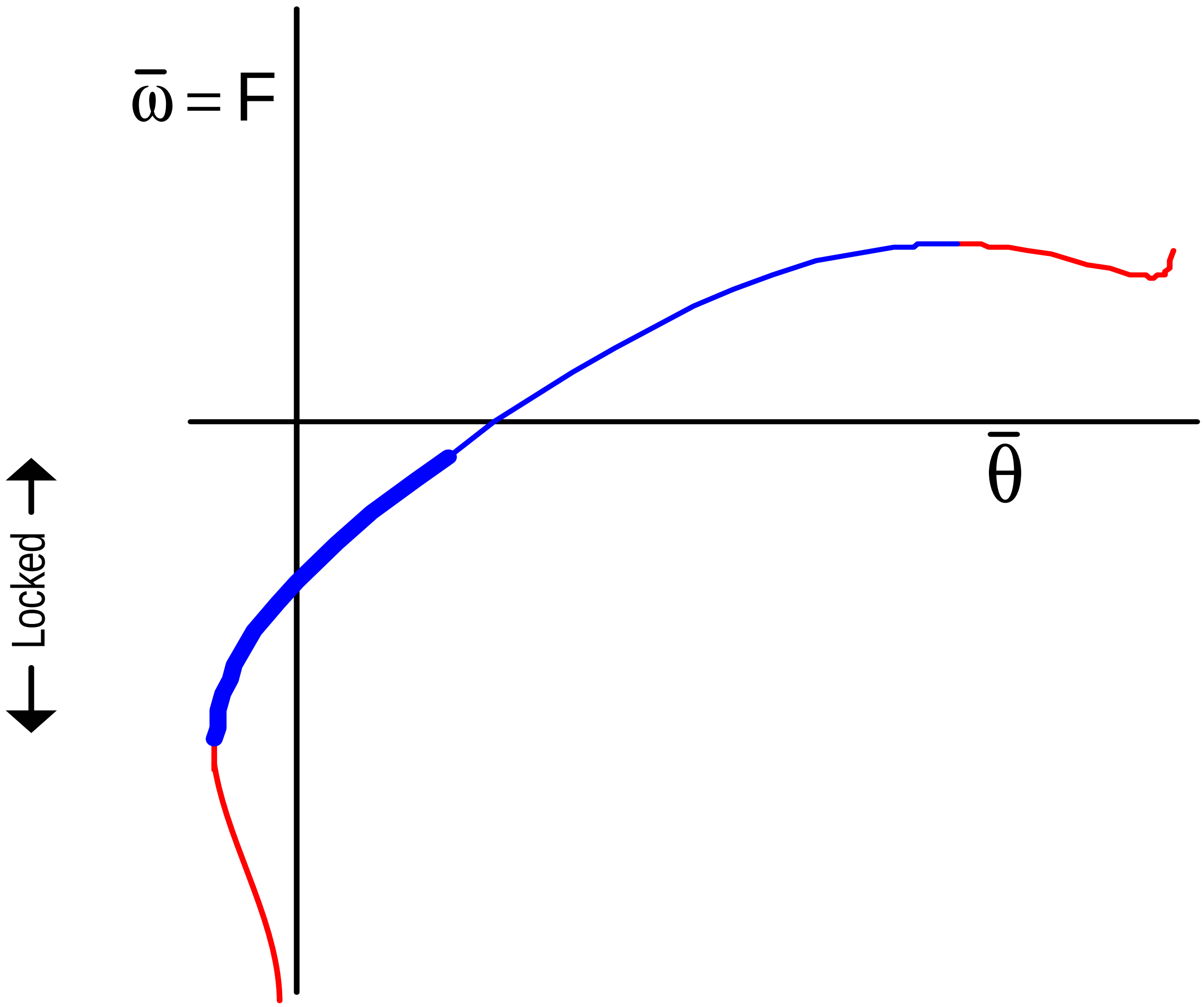




Partial locking



Full locking



Full locking

Self-consistency

$$R = \sum r_n e^{i\bar{\theta}_n}$$

- transverse

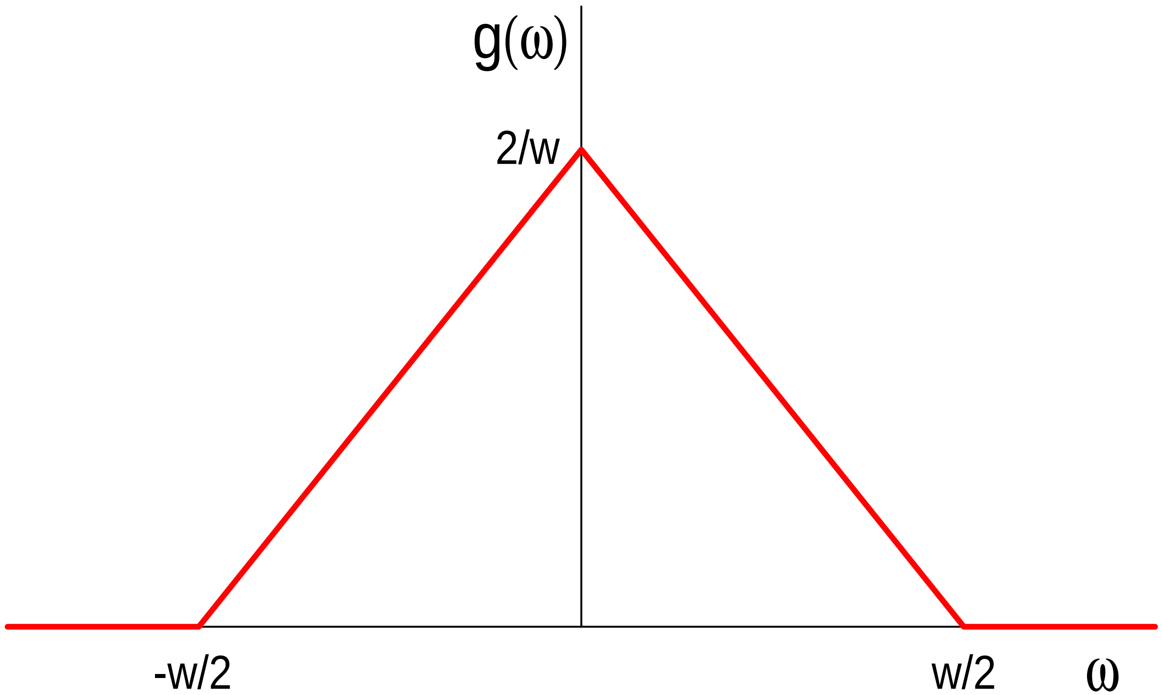
$$\int g(\bar{\omega}) r(\bar{\omega}) \sin[\bar{\theta}(\bar{\omega})] \, d\bar{\omega} = 0$$

  —determines order parameter frequency $\dot{\Theta}$ which is not in general the mean of the oscillator frequencies

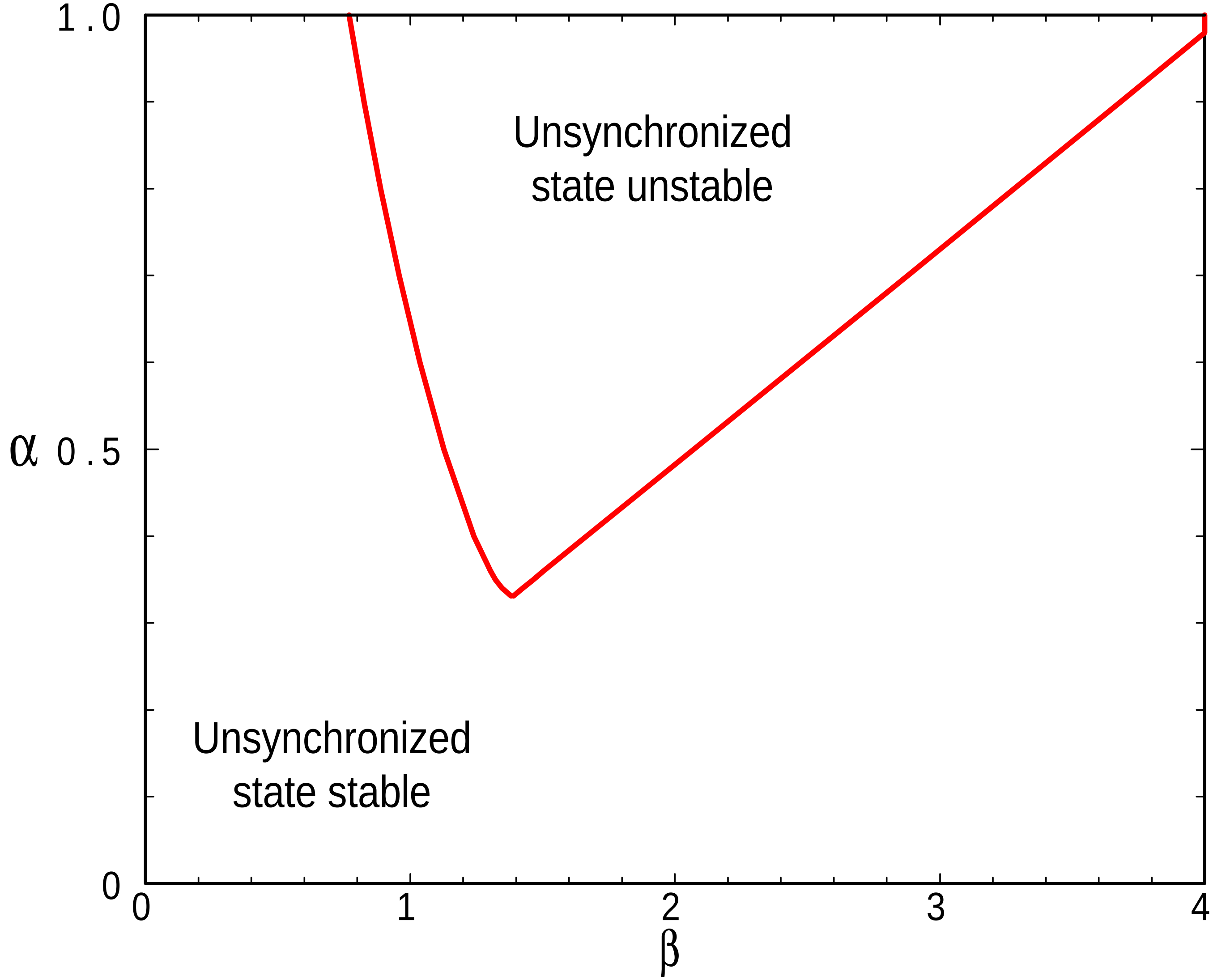
- longitudinal

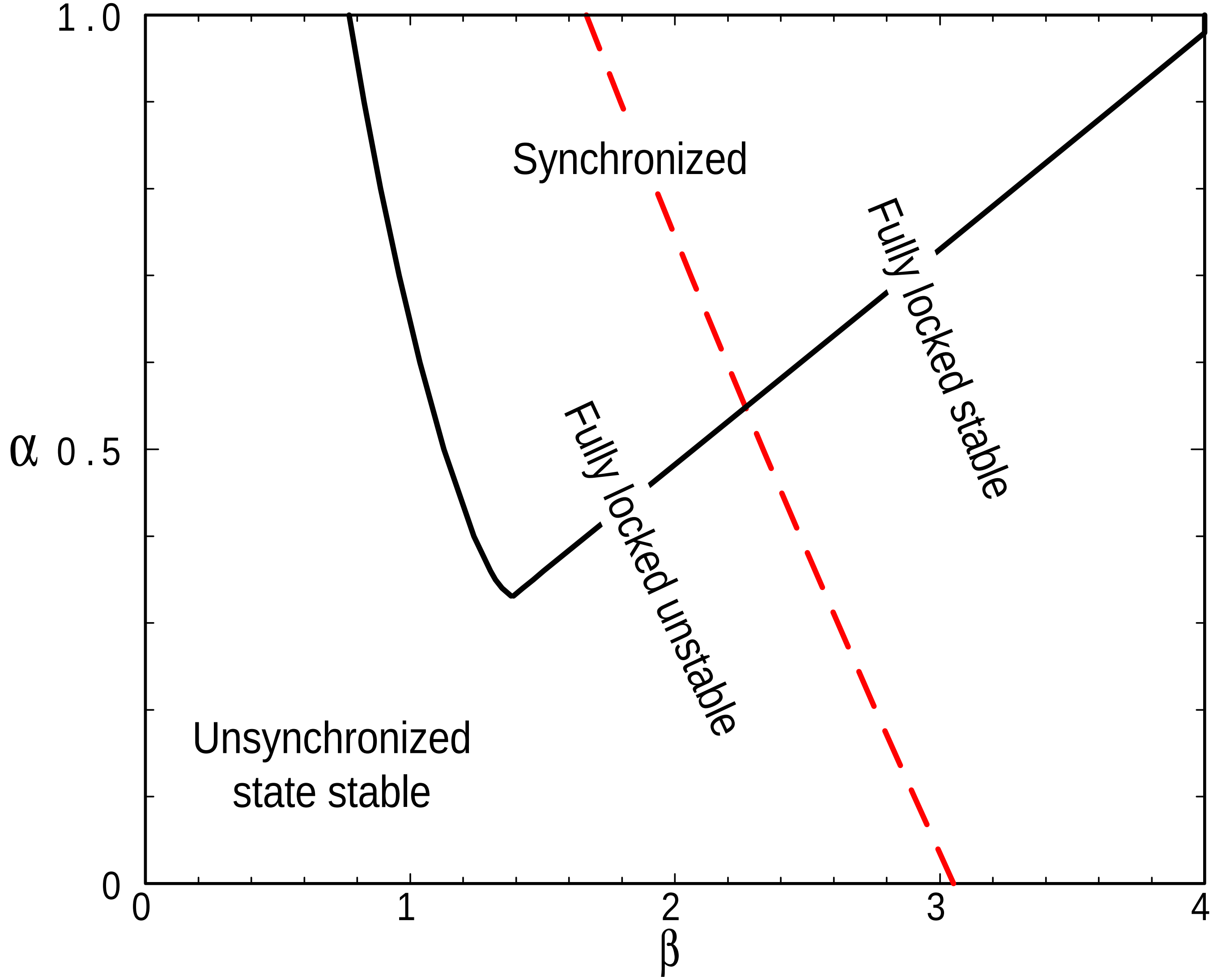
$$\int g(\bar{\omega}) r(\bar{\omega}) \cos[\bar{\theta}(\bar{\omega})] \, d\bar{\omega} = R$$

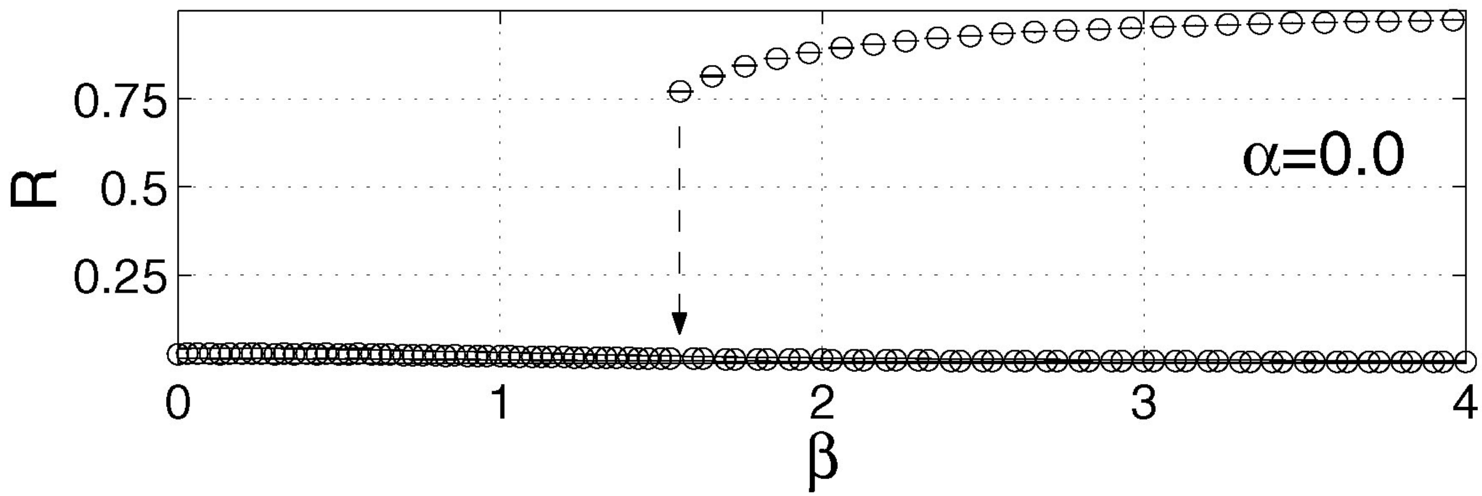
Results for a triangular distribution



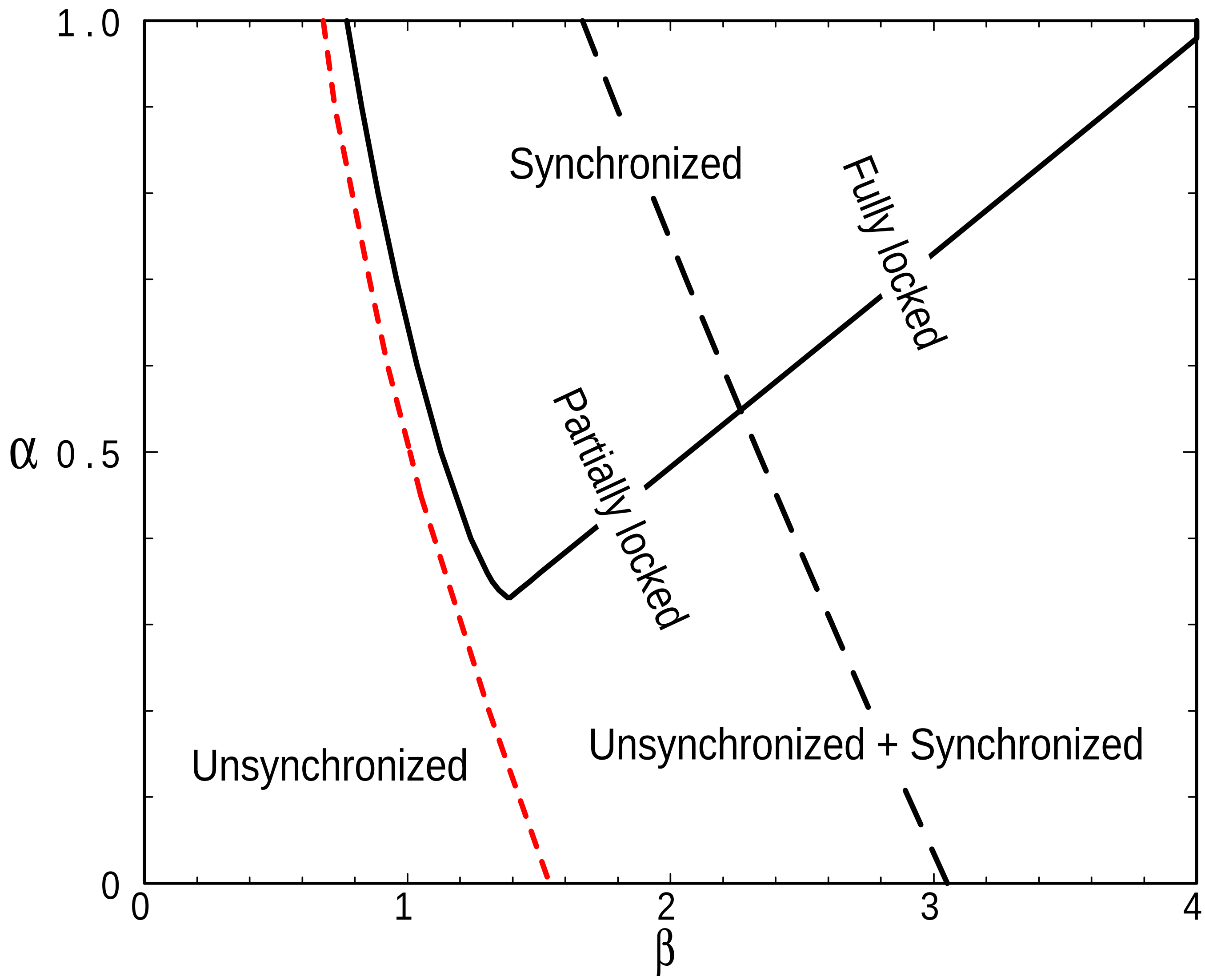
Show results for $w = 2\ldots$

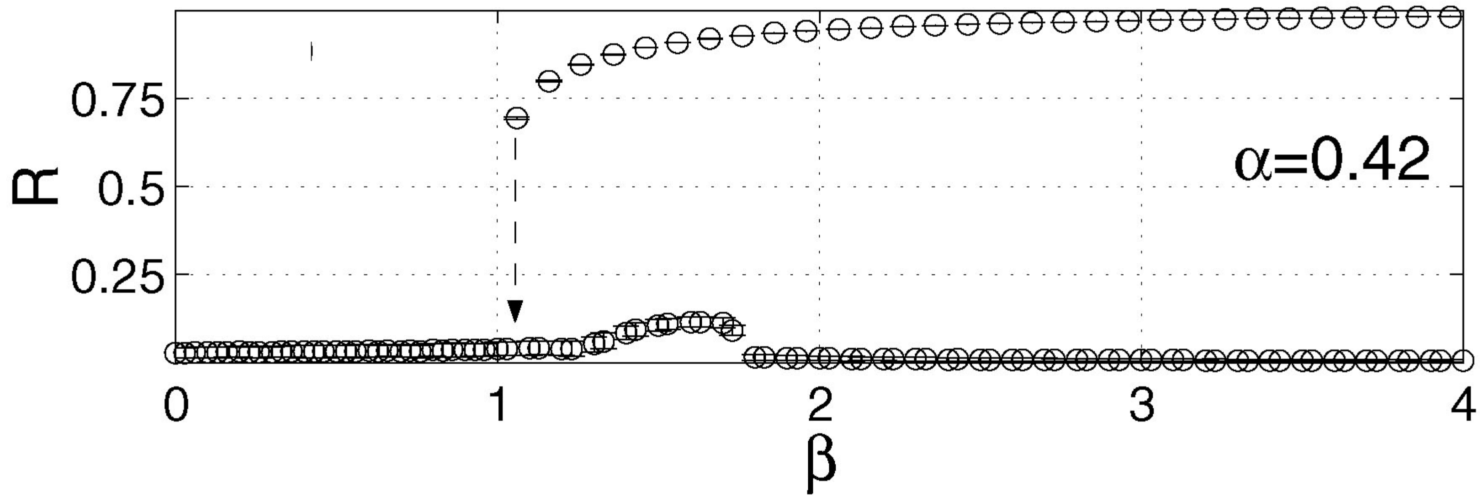

1.0
α 0.5
0
0
1
2
3
4
β
Unsynchronized
state unstable
Unsynchronized
state stable


1.0
α 0.5
0
0
1
2
3
4
β
Synchronized
Fully locked stable
Fully locked unstable
Unsynchronized
state stable

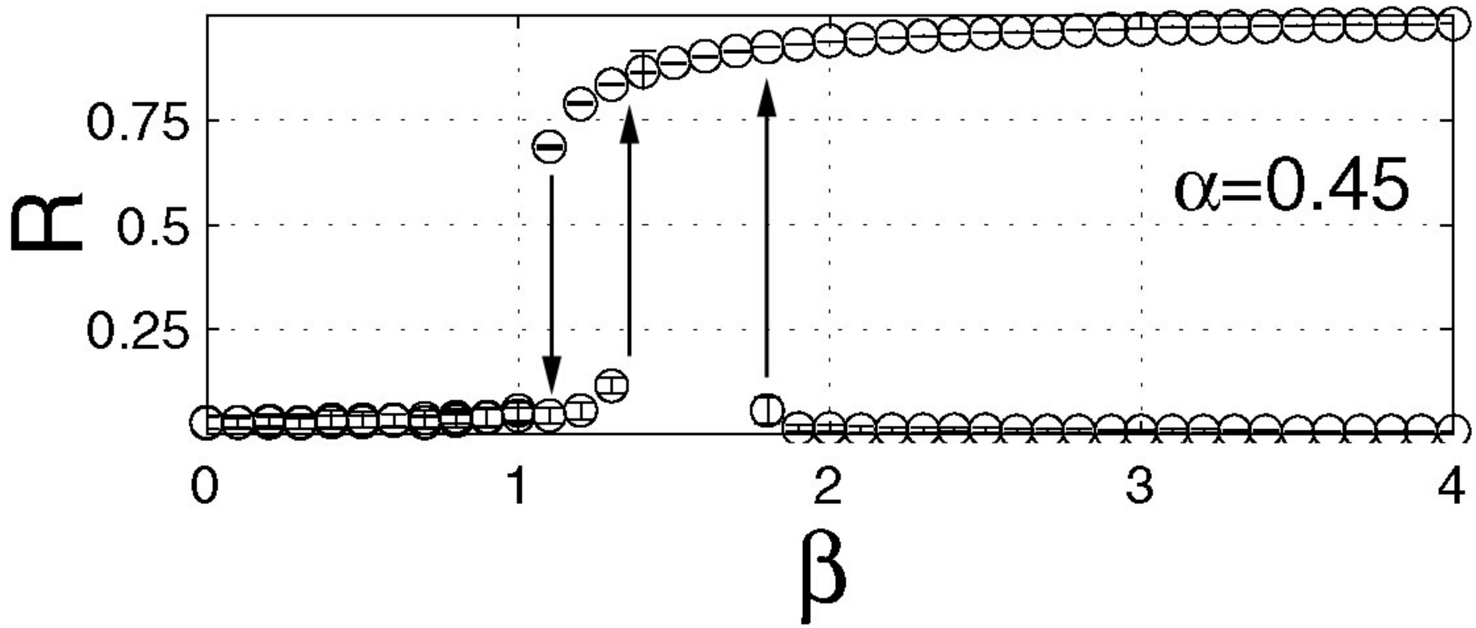


R
0.75
0.5
0.25
0
1
2
3
4
β
α=0.0

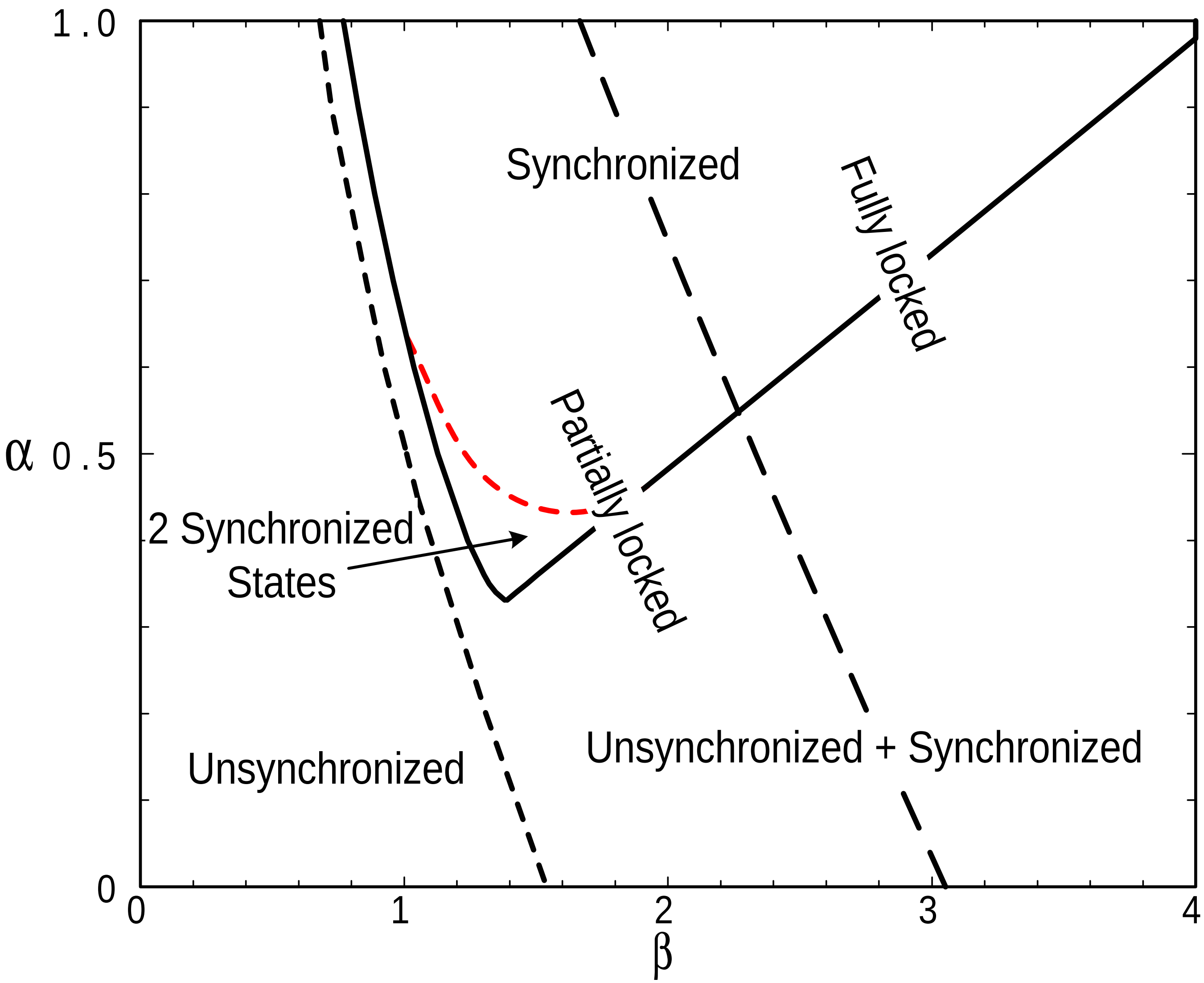

1.0
α 0.5
0
0
1
2
3
4
β
Synchronized
Fully locked
Partially locked
Unsynchronized
Unsynchronized + Synchronized

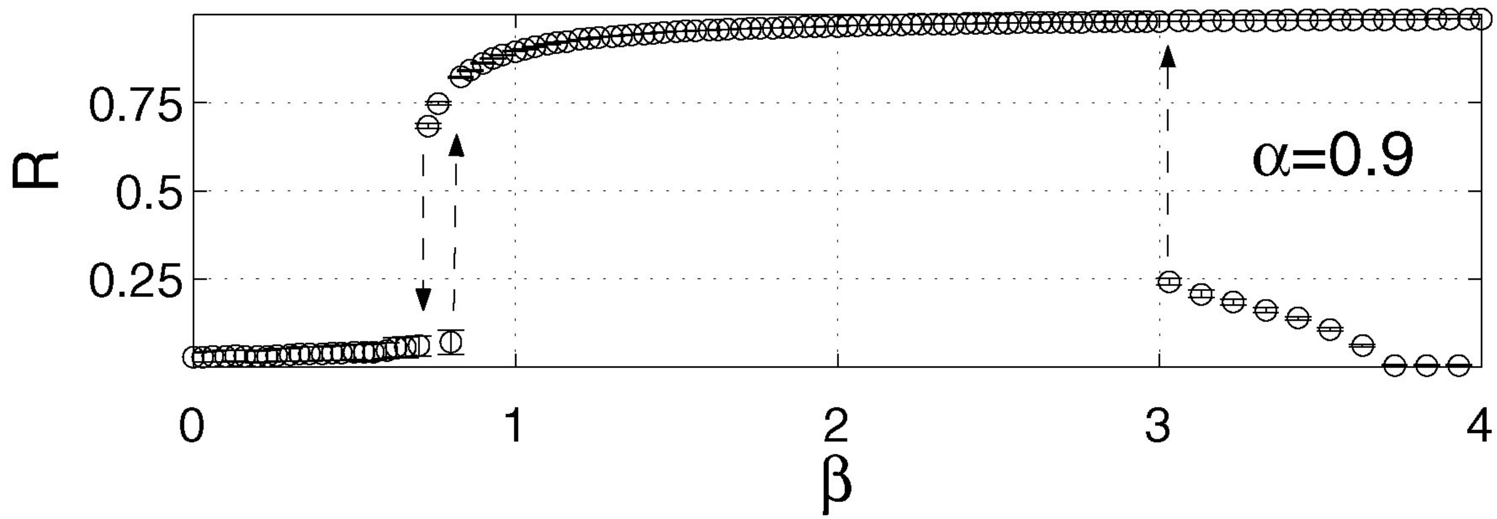


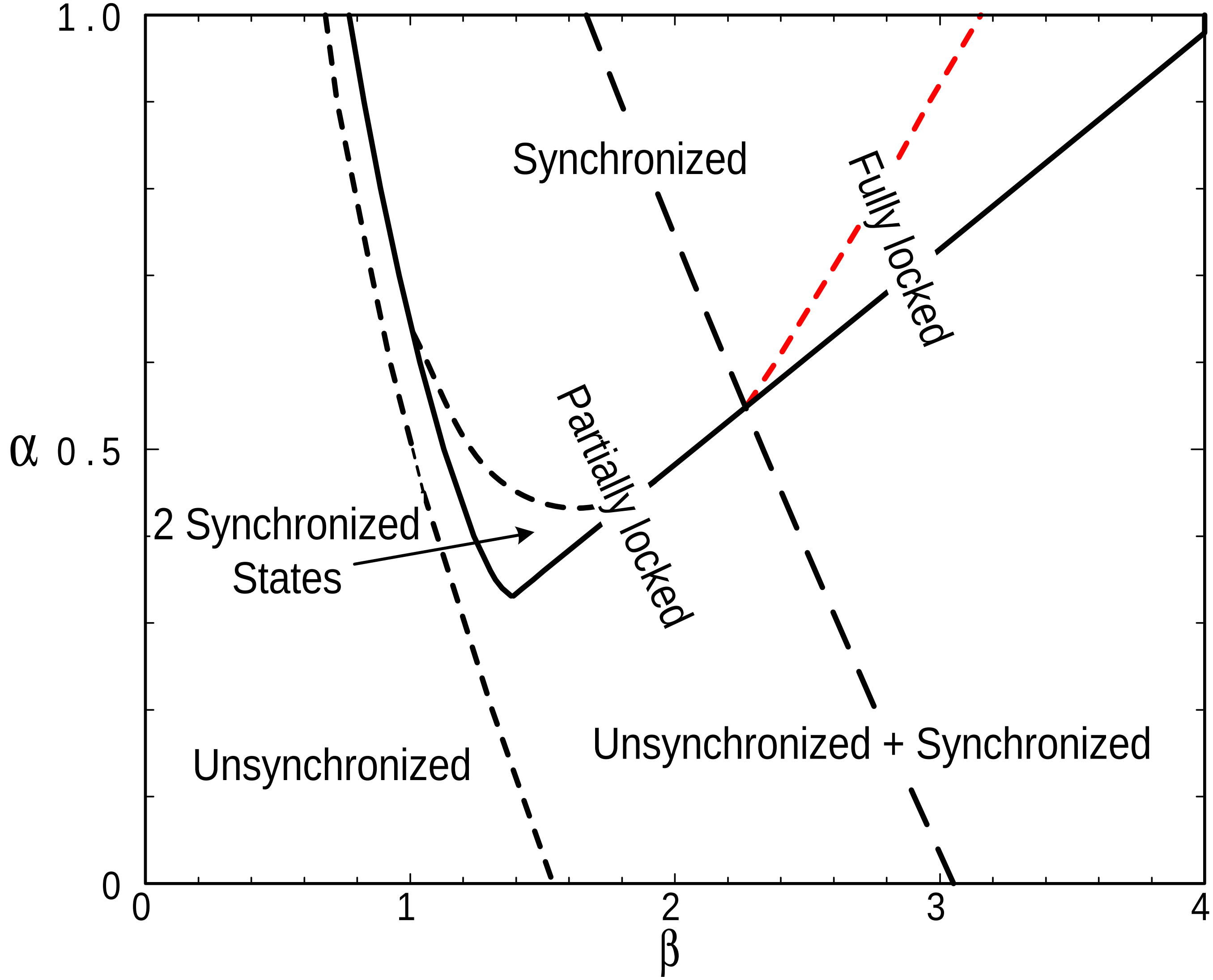
R
0.75
0.5
0.25
0
1
2
3
4
β
α=0.42

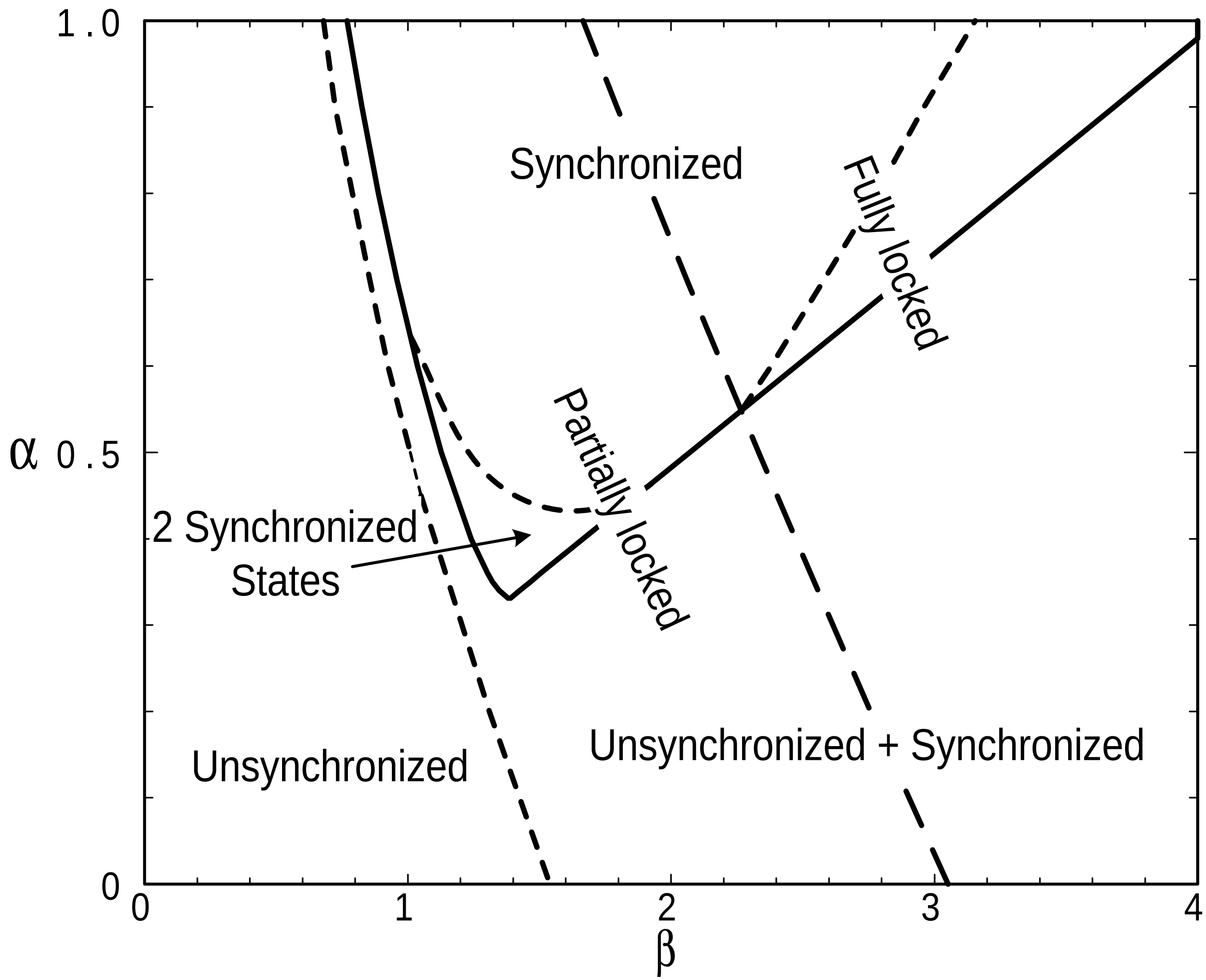


R
0.75
0.5
0.25
0
1
2
3
4
β
α=0.45


1.0
α 0.5
0
0
1
2
3
4
β
Synchronized
Fully locked
Partially locked
2 Synchronized
States
Unsynchronized
Unsynchronized + Synchronized


R
0.75
0.5
0.25
0
1
2
3
4
β
α=0.9


1.0
α 0.5
0
0
1
2
3
4
β
Synchronized
Fully locked
Partially locked
2 Synchronized
States
Unsynchronized
Unsynchronized + Synchronized


1.0
α 0.5
0
0
1
2
3
4
β
Synchronized
Fully locked
Partially locked
2 Synchronized
States
Unsynchronized
Unsynchronized + Synchronized

Conclusions

Micromechanical devices suggests a model for synchronization due to reactive coupling and nonlinear frequency pulling

Linear stability results for unsynchronized state (triangular, tophat, and Lorentzian distributions) and fully locked state (triangular, tophat)

Together with simulations yields a rich phase diagram of synchronized behavior

Novel state that is "synchronized" $R > 0$ but not frequency locked (no oscillator with $\dot{\theta} = \dot{\Theta}$)

Derivation of equation for amplitude $A_n(T)$

$$\dot{x}_n = \left([iA_n + \epsilon A_n']e^{it} + c.c. \right) + \epsilon \dot{x}_n^{(1)}(t) + \ldots$$

$$\ddot{x}_n = \left([-A_n + 2i\epsilon A_n' + \epsilon^2 A_n'']e^{it} + c.c. \right) + \epsilon \ddot{x}_n^{(1)}(t) + \ldots$$

At $O(\varepsilon)$

$$\ddot{x}_n^{(1)} + x_n^{(1)} = -\bar{\omega}_n A_n e^{it} - \left(2iA_n' e^{it} + c.c. \right)$$
$$+ \bar{\nu} \left(iA_n e^{it} + c.c. \right) \left[1 - \left(A_n e^{it} + c.c. \right)^2 \right]$$
$$+ \bar{a} \left(A_n e^{it} + c.c. \right)^3 - \bar{D}[\left(A_n - \tfrac{1}{2}(A_n + A_{n-1}) \right) e^{it} + c.c.]$$

To eliminate secular terms in e^{it}

$$2A_n' = (\bar{\nu} + i\bar{\omega}_n)A_n - (\bar{\nu} + 3i\bar{a}) |A_n|^2 A_n - i\bar{D}[A_n - \tfrac{1}{2}(A_{n+1} + A_{n-1})] = 0$$