Physics 127c: Statistical Mechanics

Superconductivity: Microscopics

For repulsive interactions the properties of an interacting Fermi system are not qualitatively different from the noninteracting system: the quantitative values of parameters are modified as described by Fermi liquid theory. Attractive interactions however, no matter how weak, lead to an entirely new state of superconductivity. It took almost 50 years from the discovery by Kammerlingh-Onnes in 1911 to the BCS theory by Bardeen, Cooper and Schrieffer in 1956 for this remarkable new state of matter to be understood.

The Cooper Problem

A simple indication of the strange consequences of attractive interactions added to the Fermi gas was demonstrated by Cooper [Phys. Rev. **104**, 1189 (1956)].

First consider the familiar problem of pair binding by free particles with an attractive pair interaction $u(r)$, but in a momentum representation. Schrodinger's equation for the problem is

$$\left[\frac{-\hbar^2}{2\mu} \nabla^2 + u(r) \right] \phi(\mathbf{r}) = E\phi(\mathbf{r}) \tag{1}$$

with $\mu = m/2$ the reduced mass and $\phi(\mathbf{r})$ the wave function of the relative coordinate $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. Introduce the Fourier representation

$$\phi(\mathbf{r}) = \sum_{\mathbf{k}'} \phi_{\mathbf{k}'} e^{i\mathbf{k}' \cdot \mathbf{r}} = \sum_{\mathbf{k}'} \phi_{\mathbf{k}'} e^{i\mathbf{k}' \cdot (\mathbf{r}_1 - \mathbf{r}_2)}, \tag{2}$$

substitute, multiply through by $e^{-i\mathbf{k} \cdot \mathbf{r}}$ and integrate over the volume V gives

$$(2\varepsilon_k - E)\phi_{\mathbf{k}} + \frac{1}{V} \sum_{\mathbf{k}'} \tilde{u}(\mathbf{k}, \mathbf{k}')\phi_{\mathbf{k}'} = 0, \tag{3}$$

with $\varepsilon_k = \hbar^2 k^2 / 2m$ and

$$\tilde{u}(\mathbf{k}, \mathbf{k}') = \tilde{u}(\mathbf{k} - \mathbf{k}') = \int u(\mathbf{r}) e^{-i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}}. \tag{4}$$

Cooper considered the following problem. Imagine two particles interacting with each other above a sea of states $k < k_F$ that are excluded from participation. The sea of states is meant to represent the Fermi sea, and the two particles cannot scatter into these states by the exclusion principle. In this case the wave function must be constructed of states with $k > k_F$

$$\phi(\mathbf{r}) = \sum_{\substack{\mathbf{k}' \\ k' > k_F}} \phi_{\mathbf{k}'} e^{i\mathbf{k}' \cdot \mathbf{r}}, \tag{5}$$

and the sum in Eq. (3) is also restricted to $k' > k_F$. To make the calculation tractable, Cooper assumed a simple attractive *separable, band limited* potential

$$\tilde{u}(\mathbf{k}, \mathbf{k}') = \begin{cases} -g & k_F < k, k' < k_F + k_c \\ 0 & \text{otherwise} \end{cases}, \tag{6}$$

with g the coupling constant. Equation (3) now becomes

$$(2\varepsilon_k - E)\phi_{\mathbf{k}} - g\frac{1}{V} \sum_{\text{band}} \phi_{\mathbf{k}'} = 0. \tag{7}$$

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We are interested in the question of whether a bound state forms with total energy $E < 2\varepsilon_F$, and so it is convenient to measure energies with respect to the Fermi energy

$$\xi_k = \varepsilon_k - \varepsilon_F \tag{8}$$

and define the binding energy E_B by $E = 2\varepsilon_F - E_B$ (then E_B is positive for a bound state). This gives

$$(2\xi_k + E_B)\phi_{\mathbf{k}} - g\frac{1}{V}\sum_{\text{band}}\phi_{\mathbf{k'}} = 0. \tag{9}$$

Clearly the solution is

$$\phi_{\mathbf{k}} = \frac{A}{(2\xi_k + E_B)} \tag{10}$$

and substituting back gives the eigenvalue equation

$$1 = g\frac{1}{V}\sum_{\text{band}}\frac{1}{(2\xi_k + E_B)}. \tag{11}$$

Replacing the sum over wave vectors by an integral over energy states gives

$$1 = gN(0)\int_0^{\hbar\omega_c}\frac{1}{(2\xi + E_B)}d\xi \tag{12}$$

with $\omega_c = v_F k_c$ the cutoff frequency and $N(0)$ the density of states of one spin system at the Fermi surface [1]. The integral is a log, giving

$$E_B = \frac{2\omega_c}{e^{2/N(0)g} - 1} \simeq 2\omega_c e^{-2/N(0)g} \tag{13}$$

where I have assumed the *weak coupling limit* $N(0)g \ll 1$ (roughly, interaction potential much less than the Fermi energy).

The expression (13) for the binding energy provides interesting insights. There is *always* a bound state, no matter how weak the attractive interaction. The binding energy dependence on the coupling constant g is nonanalytic—an essential singularity as $g \to 0$. These results are analogous to pair binding of free particles in two spatial dimension, and indeed the particles effectively "skate" on the two dimensional Fermi surface. Finally the wave function

$$\phi(\mathbf{r}) \propto \sum_{\text{band}}\frac{1}{(2\xi_k + E_B)}e^{i\mathbf{k}\cdot\mathbf{r}} \tag{14}$$

is the superposition of plane waves with $\mathbf{k}$ in a band of wave numbers of width $E_B/\hbar v_F$ near k_F. So the wave function will oscillate with a wavelength of order k_F^{-1} and will decay on a much longer length of order $\hbar v_F/E_B$. If we suppose E_B sets the energy scale of the superconducting state, and so can be estimated as $k_B T_c$ with T_c the transition temperature to the superconducting state, the pair radius is of order $(\varepsilon_F/k_B T_c)k_F^{-1}$, much larger than the interparticle spacing k_F^{-1} since $k_B T_C \ll \varepsilon_F$.

The wave function $\phi(\mathbf{r}_1 - \mathbf{r}_2)$ is symmetric under the exchange of particles, and so the spin state of the pair has to be the antisymmetric singlet $1/\sqrt{2}(\uparrow\downarrow - \downarrow\uparrow)$.

Many of the features of the solution to the Cooper problem survive in the full treatment. However the calculation is inconsistent, since the two particles are excluded from the Fermi sea because they are indistinguishable from the particles there, but we have supposed a different interaction term (none) with these. Adding this interaction means that the pair under focus will excite other particle-hole pairs, so we must consider the many body problem of many interacting particles and holes with k near k_F. This is the problem BCS solved. But first it is interesting to ask: Where does the attractive interaction come from?

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[1]A notation confusion: in the original solution set to homework 2 the TA used $N(0)$ as the *total* density of states at the Fermi surface—two times my $N(0)$. I have changed this in a revised version, but if you have the original version you should be aware of this when you make a comparison.

Attractive Interaction

See Ashcroft and Mermin §26.

BCS Theory

The BCS approach can be motivated in terms of a Bose condensed pair wave function variational ansatz

$$\Psi \propto \mathcal{A}\left[\phi(\mathbf{r}_1 - \mathbf{r}_2; \sigma_1\sigma_2)\phi(\mathbf{r}_3 - \mathbf{r}_4; \sigma_3\sigma_4)\dots\phi(\mathbf{r}_{N-1} - \mathbf{r}_N; \sigma_{N-1}\sigma_N)\right] \tag{15}$$

with $\mathcal{A}$ the antisymmetrization operator. For conventional superconductors the *pair wave function* ϕ is an s-wave, spin singlet state, and I will focus on this case. In superfluid He^3 ϕ is a p-wave, spin triplet state, and in high-T_c superconductors a d-wave, spin singlet state.

The Fourier representation for the s-wave singlet state is

$$\phi(\mathbf{r}_1 - \mathbf{r}_2; \sigma_1\sigma_2) = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)} \chi(k) \frac{1}{\sqrt{2}}(\uparrow_1\downarrow_2 - \downarrow_1\uparrow_2) \tag{16}$$

which we can write as the state

$$|\phi\rangle = \sum_{\mathbf{k}} \frac{1}{\sqrt{2}} \chi(k)\left[(\mathbf{k}\uparrow)_1(-\mathbf{k}\downarrow)_2 - (-\mathbf{k}\downarrow)_1(\mathbf{k}\uparrow)_2\right] \tag{17}$$

(using $\mathbf{k} \to -\mathbf{k}$ in the second term). This shows that in the wave function Ψ the states $(\mathbf{k}\uparrow, -\mathbf{k}\downarrow)$ are always occupied together or are empty together.

Keeping track of the amplitudes of the different combinations of the $(\mathbf{k}\uparrow, -\mathbf{k}\downarrow)$ states in Ψ is very complicated. BCS theory is equivalent to the assumption of a product state in Fourier representation

$$\Psi_{BCS} = \prod_{\mathbf{k}} \phi_{\mathbf{k}}, \tag{18}$$

with

$$\phi_{\mathbf{k}} = u_k |0, 0\rangle + v_k |1, 1\rangle, \tag{19}$$

for occupation of the $\mathbf{k}\uparrow$ and $-\mathbf{k}\downarrow$ states. Here u_k, v_k are functions to be found (with $|u_k|^2 + |v_k|^2 = 1$ by normalization). The assumption of s-wave pairing means they are functions of $|\mathbf{k}|$ only. To actually do the manipulations it is often useful to go to second quantized representation. In this notation Ψ_{BCS} is

$$\Psi_{BCS} = \prod_{\mathbf{k}} (u_k + v_k a^+_{\mathbf{k}\uparrow} a^+_{-\mathbf{k}\downarrow}) |0\rangle, \tag{20}$$

with $|0\rangle$ the no-particle or vacuum state.

To find u_k, v_k minimize $E - \mu N$ for this trial wave function. The kinetic energy relative to $N\mu$ is

$$\langle E_{\text{kin}} - \mu N\rangle = \sum_{\mathbf{k}} 2\xi_k |v_k|^2, \tag{21}$$

since there is probability $|v_k|^2$ of both $\mathbf{k}\uparrow$ and $-\mathbf{k}\downarrow$ to be occupied. The potential energy comes from sums of terms with matrix elements for the potential component $\tilde{u}(\mathbf{k}, \mathbf{k}')$ scattering a pair from occupied states $(\mathbf{k}\uparrow, -\mathbf{k}\downarrow)$ to empty states $(\mathbf{k}'\uparrow, -\mathbf{k}'\downarrow)$. The initial state has amplitude $v_k u_{k'}$ and the final state has amplitude $u_k v_{k'}$ so that the potential given by the sum of such terms is

$$\langle E_{\text{pot}}\rangle = \frac{1}{V} \sum_{\mathbf{k}, \mathbf{k}'} \tilde{u}(\mathbf{k}, \mathbf{k}') u_k^* v_{k'}^* u_{k'} v_k. \tag{22}$$

This is the right answer, but to make sure the numerical factors are right it is more reliable to take the expectation value of the second quantized version

$$U = \frac{1}{2V} \sum_{\mathbf{k},\mathbf{k}'} \tilde{u}(\mathbf{q}) a^+_{\mathbf{k}+\mathbf{q},\sigma} a_{\mathbf{k}'-\mathbf{q},\sigma'} a_{\mathbf{k}',\sigma'} a_{\mathbf{k},\sigma} \tag{23}$$

in the state (20).

Thus

$$\langle E - \mu N \rangle = \sum_{\mathbf{k}} 2\xi_k |v_k|^2 + \frac{1}{V} \sum_{\mathbf{k},\mathbf{k}'} \tilde{u}(\mathbf{k}, \mathbf{k}') u_k^* v_{k'}^* u_{k'} v_k. \tag{24}$$

This equation couples the phases of the u_k, v_k at different $\mathbf{k}$. For s-wave paring there is an overall phase factor (which wold be the phase of the order parameter) which can be set to zero in calculating the energetics. Thus we may assume u_k, v_k are real. For p- and d- wave pairing we would be dealing with functions $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ also depending on the direction of $\mathbf{k}$, and there would be nontrivial phases depending on this direction corresponding to the phase of the $l = 1, 2$ pair wave functions. From the normalization condition we can write for the s-wave case

$$u_k = \sin \theta_k, \qquad v_k = \cos \theta_k \tag{25}$$

giving

$$\langle E - \mu N \rangle = \sum_{\mathbf{k}} 2\xi_k (1 + \cos 2\theta_k) + \frac{1}{V} \sum_{\mathbf{k},\mathbf{k}'} \tilde{u}(\mathbf{k}, \mathbf{k}') \frac{1}{4} \sin 2\theta_k \sin 2\theta_{k'} \tag{26}$$

with θ_k given by minimizing this energy. Minimizing gives

$$\tan 2\theta_k = \frac{1}{2\xi_k} \frac{1}{V} \sum_{\mathbf{k}'} \tilde{u}(\mathbf{k}, \mathbf{k}') \sin 2\theta_{k'}. \tag{27}$$

Define the *gap function*

$$\Delta_k = -\frac{1}{2V} \sum_{\mathbf{k}'} \tilde{u}(\mathbf{k}, \mathbf{k}') \sin 2\theta_{k'}, \tag{28}$$

and the function that will turn out to be the excitation energy

$$E_k = \sqrt{\xi_k^2 + \Delta_k^2}. \tag{29}$$

Then

$$\tan 2\theta_k = -\frac{\Delta_k}{\xi_k}, \qquad \sin 2\theta_k = \frac{\Delta_k}{E_k}, \tag{30}$$

and

$$u_k v_k = \frac{\Delta_k}{2E_k}, \quad v_k^2 = \frac{1}{2}(1 - \frac{\xi_k}{E_k}), \quad u_k^2 = \frac{1}{2}(1 + \frac{\xi_k}{E_k}) \tag{31}$$

(Note that $v_k^2 \to 1$ for $k \ll k_F$, $\xi_k/E_k \to -1$, and $v_k^2 \to 0$ for $k \gg k_F$, $\xi_k/E_k \to 1$ as required: thus the sign chosen for E_k is correct.)

Equation (28) becomes

$$\Delta_k = -\frac{1}{V} \sum_{\mathbf{k}'} \tilde{u}(\mathbf{k}, \mathbf{k}') \frac{\Delta_{k'}}{2E_{k'}}, \tag{32}$$

which is a self consistency condition for the gap parameter $\Delta_{\mathbf{k}}$.

Equation (32) defines the gap parameter Δ_k for given interaction potential. Again to make analytic progress it is useful to consider the simple model of the separable potential, now changed to be symmetric about k_F

$$\tilde{u}(\mathbf{k}, \mathbf{k}') = \begin{cases} -g & k_F - k_c < k, k' < k_F + k_c \\ 0 & \text{otherwise} \end{cases} . \tag{33}$$

This gives

$$\Delta_k = \begin{cases} \Delta_0 & k_F - k_c < k, k' < k_F + k_c \\ 0 & \text{otherwise} \end{cases} . \tag{34}$$

Transforming to an integration over energy for the spherically symmetric s-wave case the single parameter Δ_0 is given by

$$1 = N(0)g \int_0^{\omega_c} \frac{1}{\sqrt{\xi^2 + \Delta_0^2}} \tag{35}$$

$$= N(0)g \sinh^{-1}(\hbar\omega_c/\Delta), \tag{36}$$

or

$$\Delta = \frac{\hbar\omega_c}{\sinh^{-1}(\hbar\omega_c/\Delta)} \simeq 2\hbar\omega_c e^{-1/N(0)g}, \tag{37}$$

using weak coupling in the last approximation. It is straightforward to check that $E - \mu N$ is lowered for this value of Δ relative to the normal state $\Delta = 0$.

To compare the results of this calculation with the Cooper problem it is useful to look at the average occupation number $\langle n_k \rangle = v_k^2$. The step function at $k = k_F$ in v_k for the noninteracting problem is spread into a smooth variation over a width in $k \sim \Delta/\hbar v_F$ (a width in energy of about Δ). This corresponds to the self consistent excitation of pairs out of the Fermi sea to gain the Cooper-type pairing energy.
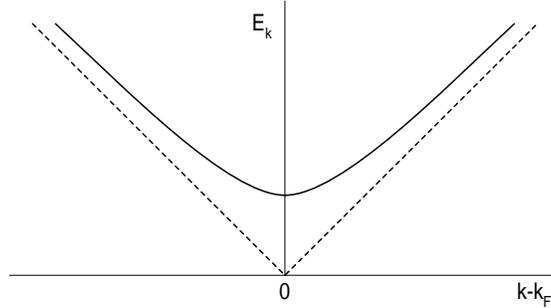

Figure 1: Excitation energy E_k for a BCS superconductor (solid line) and normal state (dashed line). The energy is defined as the cost to add a particle to a state $\mathbf{k}$ with $k > k_F$ relative to the chemcial potential or to remove a particle from $\mathbf{k}$ with $k < k_F$ relative to $-\mu$. Thus the normal state spectrum is $v_F |k - k_F|$.

To see the significance of Δ and E_k we look at the excited states of the system. For the states $\mathbf{k} \uparrow, -\mathbf{k} \downarrow$ in the product wave function the four eigenstates and energies measured with resect to the pair state are

	State	Energy
pair	$u_k \lvert 0, 0 \rangle + v_k \lvert 1, 1 \rangle$	0
broken pair	$\lvert 1, 0 \rangle$	E_k
broken pair	$\lvert 0, 1 \rangle$	E_k
excited state	$v_k \lvert 0, 0 \rangle - u_k \lvert 1, 1 \rangle$	$2E_k$

(38)

Consider the broken pair state $|1, 0\rangle$ which is the state with $\mathbf{k} \uparrow$ occupied and $-\mathbf{k} \downarrow$ empty. The contribution to kinetic energy $E_{\text{kin}} - N\mu$ is ξ_k whereas in the pair state it is, from Eq. (21), $\xi_k(1 - \xi_k/E_k)$. In the pairing energy Eq. (22) the state $\mathbf{k}$ is removed from the both wave vector sums, and so the pairing energy is reduced by

$$\frac{2}{V} \sum_{\mathbf{k}'} \tilde{u}(\mathbf{k}, \mathbf{k}') u_k v_{k'} u_{k'} v_k = \frac{\Delta_k}{E_k} \sum_{\mathbf{k}'} \tilde{u}(\mathbf{k}, \mathbf{k}') \frac{\Delta_{k'}}{2E_{k'}} = -\frac{\Delta_k^2}{E_k}, \tag{39}$$

and so the excitation energy is

$$\xi_k - \xi_k \left(1 - \frac{\xi_k}{E_k}\right) + \frac{\Delta_k^2}{E_k} = E_k. \tag{40}$$

For the excited pair state the change in the kinetic energy is

$$\xi_k \left(1 + \frac{\xi_k}{E_k}\right) - \xi_k \left(1 - \frac{\xi_k}{E_k}\right). \tag{41}$$

In the pairing energy calculation of the amplitude for scattering involving the pair state $\mathbf{k} \uparrow, -\mathbf{k} \downarrow$, the product of amplitudes that the state is occupied before scattering and empty after scattering is $-v_k u_k$ instead of $u_k v_k$. Thus the contribution to the pairing energy is the negative of what it was for the ground state pair. This gives the excitation energy

$$\xi_k \left(1 + \frac{\xi_k}{E_k}\right) - \xi_k \left(1 - \frac{\xi_k}{E_k}\right) + \frac{2\Delta_k^2}{E_k} = 2E_k. \tag{42}$$

Thus E_k plays the role of the excitation energy: to add a particle in the excited state costs an energy $E_k = \sqrt{\xi_k^2 + \Delta_k^2}$. with $\Delta_k = \Delta$ for $|k - k_F| < k_c$. The minimum energy is for $k = k_F$, $\xi_k = 0$, showing that Δ is the *energy gap* for excitations. The minimum energy cost to break a pair to form two broken pair states, or to form the excited pair state is 2Δ, and thermodynamic quantities at low temperatures will vary as $\exp(-2\Delta/k_B T_c)$.

In Homework 2 you transformed the Hamiltonian using a canonical transformation to new independent Fermi operators α_k, β_k

$$a_{\mathbf{k}\uparrow} = u_k \alpha_{\mathbf{k}} + v_k \beta_{-\mathbf{k}}^+, \tag{43}$$

$$a_{-\mathbf{k}\downarrow} = u_k \beta_{-\mathbf{k}} - v_k \alpha_{\mathbf{k}}^+. \tag{44}$$

to the form

$$H = \text{const.} + \sum_{\mathbf{k}} E_k(\alpha_{\mathbf{k}}^+ \alpha_{\mathbf{k}} + \beta_{-\mathbf{k}}^+ \beta_{-\mathbf{k}}). \tag{45}$$

For the pair ground state $|\phi_{\mathbf{k}}\rangle = (u_k + v_k a_{\mathbf{k}\uparrow}^+ a_{-\mathbf{k}\downarrow}^+) |0\rangle$ you can show that the broken pair states corresponds to $\alpha_{\mathbf{k}}^+ |\phi_{\mathbf{k}}\rangle$ and $\beta_{-\mathbf{k}}^+ |\phi_{\mathbf{k}}\rangle$ with energy E_k and the excited pair state to $\alpha_{\mathbf{k}}^+ \beta_{-\mathbf{k}}^+ |\phi_{\mathbf{k}}\rangle$ with energy $2E_k$, showing that the results of the two calculations agree. As you will probably agree, the canonical transformation arguments are less complicated.

Thermodynamics

To calculate the finite temperature properties we could continue to enumerate the states by hand, but it is simpler to switch to the approach of Homework 2. There you found the gap equation

$$\Delta_k = \frac{1}{V} \sum_{\vec{k}'} \tilde{u}(\mathbf{k}, \mathbf{k}') \langle a_{\mathbf{k}'\uparrow} a_{-\mathbf{k}'\downarrow} \rangle \tag{46}$$

and the quantum and thermal average $\langle a_{\mathbf{k'}\uparrow} a_{-\mathbf{k'}\downarrow} \rangle$ is easy to calculate from the inverse of the canonical transformation

$$a_{\vec{k}\uparrow} = u_k \alpha_{\vec{k}} + v_k \beta^+_{-\vec{k}} \tag{47}$$

$$a_{-\vec{k}\downarrow} = u_k \beta_{-\vec{k}} - v_k \alpha^+_{\vec{k}} \tag{48}$$

and the thermal averages of the noninteracting Fermions α_k, β_k

$$\left\langle \alpha^+_{\vec{k}} \alpha_{\vec{k'}} \right\rangle = \left\langle \beta^+_{\vec{k}} \beta_{\vec{k'}} \right\rangle = f(E_k) \delta_{\vec{k}\vec{k'}} \tag{49}$$

whereas

$$0 = \left\langle \alpha^+_{\vec{k}} \beta_{\vec{k'}} \right\rangle = \left\langle \alpha^+_{\vec{k}} \beta^+_{\vec{k'}} \right\rangle = \dots \text{etc} \tag{50}$$

with $f(E_k)$ the usual Fermi function

$$f(E_k) = \frac{1}{e^{E_k/k_B T} + 1}. \tag{51}$$

The equation for the gap parameter $\Delta_k(T)$ becomes

$$\Delta_k(T) = -\frac{1}{V} \sum_{\vec{k'}} \tilde{u}(\mathbf{k}, \mathbf{k}) \frac{\Delta_{k'}(T)}{2E_{k'}} \tanh \left(\frac{E_{k'}}{2k_B T} \right), \tag{52}$$

with now $E_k = \sqrt{\xi_k^2 + \Delta_k^2(T)}$. For the same separable potential the equation for the gap $\Delta(T)$ at nonzero temperature is

$$1 = N(0)g \int_0^{\hbar \omega_c} \frac{1}{E} \tanh \left(\frac{E}{2k_B T} \right) d\xi, \tag{53}$$

with $E = \sqrt{\xi^2 + \Delta^2(T)}$. At the transition temperature T_c the superconducting gap goes to zero $\Delta \to 0$, and so T_c is given by

$$1 = N(0)g \int_0^{\hbar \omega_c} \frac{1}{\xi} \tanh \left(\frac{\xi}{2k_B T_c} \right) d\xi. \tag{54}$$

In the weak coupling limit this gives

$$k_B T_c \simeq 1.14 \hbar \omega_c e^{-1/N(0)g}. \tag{55}$$

Note that the zero temperature gap is related to T_c by

$$\frac{2\Delta_0}{k_B T_c} \simeq 3.52, \tag{56}$$

a universal result, independent of any other parameters.

The integrals in Eqs. (53) and (54) depend logarithmically on ω_c the cutoff frequency introduced in the interaction. However for the range $\xi \gg \Delta$ of the integrals contributing to this logarithmic dependence (assuming weak coupling) the integrands in Eqs. (53) and (54) are almost equal. Thus if we subtract, the contribution from large ξ vanishes, and we can replace the upper limit by ∞, to give

$$\int_0^\infty \left[\frac{1}{\sqrt{\xi^2 + \Delta^2(T)}} \tanh \left(\frac{\sqrt{\xi^2 + \Delta^2(T)}}{2k_B T} \right) - \frac{1}{\xi} \tanh \left(\frac{\xi}{2k_B T_c} \right) \right] d\xi = 0. \tag{57}$$

This gives a *universal* equation for $\Delta(T)/k_B T_c$ as a function of T/T_c. With some more effort it can be argued that this result does not in fact depend on the simple form assumed for the potential, but just on the weak coupling limit $k_B T_c \ll \hbar \omega_c$, true for any small enough attractive interaction. In this limit a universal prediction is obtained for the thermodynamics as a function of T/T_c independent of details of the potential etc.

Further Reading

You should review Homework 2 and the solution set. The original paper of Bardeen, Cooper and Schrieffer, Phys. Rev. **108**, 1175 (1956) is a classic. Not only do they present the full theory, but they work out many of the important experimental tests. The review article Rev. Mod. Phys. **47**, 331 (1975) by Leggett on superfluid He^3, also described by BCS theory but with p-wave pairs, provides a very nice discussion of BCS theory in general, particularly §III and §V, and the application to He^3 in other sections.