Physics 127b: Statistical Mechanics

Renormalization Group: General Case

The steps in the renormalization group are

1. Eliminate degrees of freedom by a scale factor b so that

$$N' = \frac{N}{b^d}, \tag{1}$$

   whilst *preserving the free energy.* This might be done by "block spinning" or integrating out high momenta components in Fourier space $\int_{K/b}^{K} q^{d-1} dq$ with K the upper momentum cutoff.

2. Rescale lengths to keep the density of degrees of freedom constant

$$x' = \frac{x}{b} \quad \Rightarrow \quad \xi' = \frac{\xi}{b}. \tag{2}$$

This gives the renormalization group

$$\bar{H}' = \mathcal{R}_b[\bar{H}]. \tag{3}$$

Under this process the free energy density changes as

$$f' = f[\bar{H}'] = b^d f[\bar{H}]. \tag{4}$$

Usually $\bar{H}$ will *not* retain the simple form of the initial Hamiltonian, e.g. if we did the same spin elimination in the d-dimensional Ising lattice as we did in 1d, we would have $(2d)^2$ possible spin states of the $2d$ nearest neighbors, and we would not be able to construct the new effective Hamiltonian with just three parameters. Instead we would have to allow more complicated interactions, such as four spin terms.

We assume that there is some *fixed point* Hamiltonian $\bar{H}^*$ *defined* by the relation

$$\bar{H}^* = \mathcal{R}_b[\bar{H}^*]. \tag{5}$$

To understand the properties for Hamiltonians near $\bar{H}^*$ we linearize about $\bar{H}^*$, i.e.

$$\bar{H} = \bar{H}^* + hQ \tag{6}$$

where Q is some correction terms to $\bar{H}^*$ (e.g. nearest neighbor interactions at the 1d Ising fixed point) and h is a small amplitude. Now under renormalization

$$\bar{H}' = \mathcal{R}_b[\bar{H}^* + hQ] = \bar{H}^* + h\mathcal{L}_b[Q] + O(h^2), \tag{7}$$

where $\mathcal{L}$ is the linearization of the renormalization group about $\bar{H}^*$. The linear operator can be "diagonalized", i.e. we find particular combinations of correction terms that grow geometrically

$$\mathcal{L}_b[Q_j] = \Lambda_j Q_j \tag{8}$$

with Q_j an eigenvector known as a *critical operator*. It is convenient to write the eigenvalue as

$$\Lambda_j = b^{\lambda_j} \tag{9}$$

and then λ_j will not depend on the choice of scale factor b.

Now expand our initial Hamiltonian in these eigenvectors

$$\bar{H} = \bar{H}^* + \sum_j h_j Q_j \tag{10}$$

and then

$$\bar{H}' = \bar{H}^* + \sum_j h_j b^{\lambda_j} Q_j + O(h^2) \tag{11}$$

so that

$$h'_j = b^{\lambda_j} h_j. \tag{12}$$

Notice that the jth field h_j will grow geometrically if λ_j is positive, but will shrink if λ_j is negative. This leads to the classification

Relevant operators	$\Lambda_j > 1$	$\lambda_j > 0$	h_j grows
Irrelevant operators	$\Lambda_j < 1$	$\lambda_j < 0$	h_j shrinks
Marginal operators	$\Lambda_j = 1$	$\lambda_j = 0$	??

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In the case of marginal operators, the $O(h^2)$ have to be investigated to decide whether the coefficient of the operator will grow or shrink under renormalization.

One of the operators Q_j will correspond somehow to the effect of changing the temperature of the interaction strength in the Hamiltonian, and we will label this one with the index zero. We order the other operators according to the size of their λ_j : $\lambda_1 > \lambda_2 > \cdots$ (i.e. decreasing "relevance" to the left).

Under iteration of the RNG

$$f(h_0, h_1, h_2 \ldots) = b^{-ld} f(b^{\lambda_o l} h_0, b^{\lambda_1 l} h_1, b^{\lambda_2 l} h_2 \ldots). \tag{13}$$

If our physical Hamiltonian happened to be near the critical one, then we would identify h_0 as t, the temperature parameter that takes us through the critical point for zero $h_{j>0}$. If t is small, and we iterate a large number of times (but not so large that $\bar{H}'$ leaves the region near $\bar{H}^*$ where linearization is a good approximation), the $b^{\lambda_j l}$ for $\lambda_j < 0$ will be sent to zero. Now, in the familiar way, as we vary our physical temperature t we choose l such that $b^{\lambda_0 l} t$ remains fixed, and find

$$f(t, h_1, h_2, \ldots) = t^{2-\alpha} W\left(\frac{h_1}{t^{\Delta_1}}, \frac{h_2}{t^{\Delta_2}} \cdots \frac{h_J}{t^{\Delta_J}}\right) \tag{14}$$

where the exponents

$$2 - \alpha = \frac{d}{\lambda_0} \quad \text{and} \quad \Delta_i = \frac{\lambda_i}{\lambda_0} \tag{15}$$

are *fixed by the eigenvalues of the linearization about $\bar{H}^*$*. We have denoted by J the last relevant field ($\lambda_i < 0$ for $i > J$).

Thus we find *scaling,* and *universality* since the exponents are determined by $\bar{H}^*$ and not by the physical system. The usual case for a second order phase transition is two relevant operators ($J = 1$): the temperature variable $h_0 = t$ and a symmetry breaking field $h_1 = h$.

The above argument gives the essential features of how the RNG leads to an understanding of scaling and universality. However there is one catch in the argument: most physical Hamiltonians will *never* pass near $\bar{H}^*$ as a single parameter (e.g. the temperature) is changed. The fact that the same basic ideas apply in this case follows from understanding the sketch in Fig. 1.
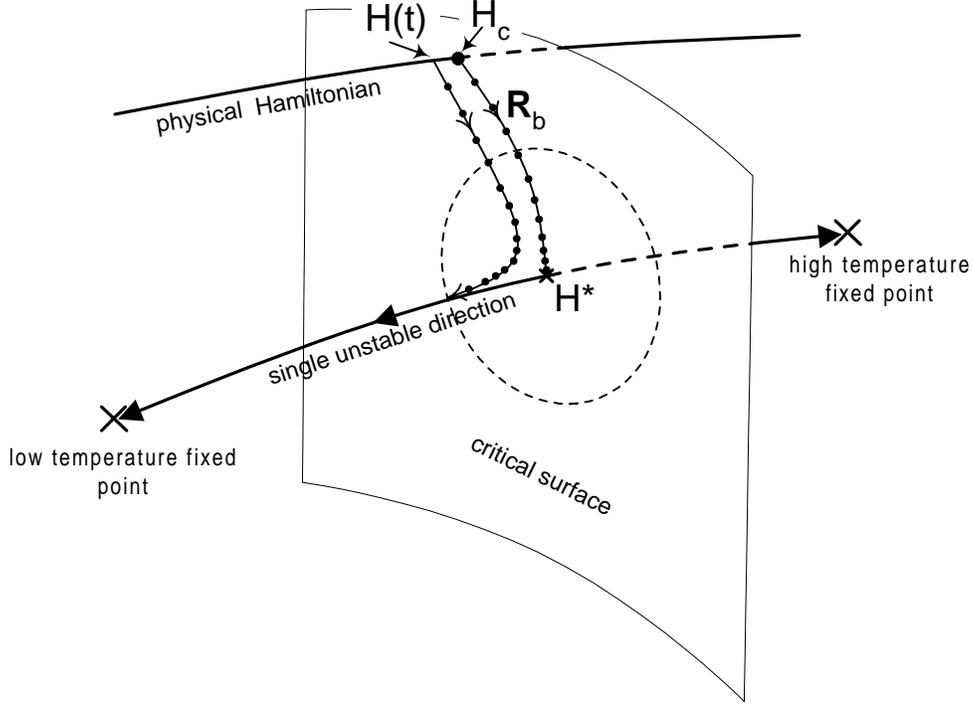
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Figure 1: Flows near the critical surface

Critical phenomena, where there are power law singularities as a function of temperature in the absence of symmetry breaking fields, occurs when there is a single relevant operator at the critical Hamiltonian $\bar{H}^*$ after setting the symmetry breaking field to zero. This means that there is a "codimension one" hypersurface in Hamiltonian space[1] such that all Hamiltonians on this hypersurface end up at $\bar{H}^*$ after sufficiently large number of iterations of $\mathcal{R}_b$

$$\mathcal{R}_b^l[\bar{H}_c] \to \bar{H}^* \quad \text{as} \quad l \to \infty. \tag{16}$$

This is the phenomenon of universality in a nutshell: all of these Hamiltonians have the same long length-scale properties as $\bar{H}^*$, and in particular correspond to systems in the universality class characterized by this $\bar{H}^*$ at their critical temperatures T_c (hence the subscript c in Eq. (16). We call this the critical surface. Notice that since this is a codimension one surface, i.e. of one dimension less than the dimension of the space) it is quite likely that as we vary a single parameter of a physical system such as the temperature, or the coupling constant, the path of Hamiltonians will intersect the critical surface. *The transition temperature T_c is defined not by $\bar{H}$ being close to $\bar{H}^*$ but by $\bar{H}$ lying on the critical surface.*

Let's understand a particular Hamiltonian $\bar{H}(t)$ that is $\bar{H}_c$ on the critical surface for $t = 0$. If we apply $\mathcal{R}_b$ to $\bar{H}_c$ some finite number of times q the Hamiltonian will evolve to be in the critical region near $\bar{H}^*$ where linearization about $\bar{H}^*$ is good. Now lets follow $\bar{H}(t)$ for t small under $\mathcal{R}_b$: since t is small the trajectory will follow that from $\bar{H}_c$, and after q iterations $\mathcal{R}_b^q[\bar{H}(t)]$ will also be in the critical region. We can then expand in the critical operators:

$$\mathcal{R}_b^q[\bar{H}(t)] = \bar{H}^* + t A_0 Q_0 + \sum_{j>0} A_j Q_j. \tag{17}$$

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[1]What the "space of all Hamiltonians" means, and what is its dimension, are hard to answer precisely. Indeed mathematicians were wary of the renormalization group until its application to the period doubling route to chaos by Mitch Feigenbaum. In that case the space in which the RNG acts is easier to define mathematically.

Notice that *the amplitude of the relevant operator is proportional to t*, since it is certainly zero for $t = 0$, and we have arrived at this point by some finite number q of iterations *independent of the size of t*. The amplitudes of the irrelevant operators are finite for $t = 0$, and it is sufficient to keep these leading order values.

Now we operate a further $l - q$ times, where l becomes large as $t \to 0$:

$$\mathcal{R}_b^l[\bar{H}(t)] = \mathcal{R}_b^{l-q}[\bar{H}^* + tA_0Q_0 + \sum_{j>0} A_jQ_j] \tag{18}$$

$$\simeq \bar{H}^* + tA_0b^{\lambda_0(l-q)}Q_0 \tag{19}$$

since the irrelevant components shrink to zero for large l. For the free energy we have

$$f[\bar{H}(t)] = b^{-ld}f[\bar{H}^* + tb^{\lambda_0 l}(A_0b^{-\lambda_0 q})Q_0] \tag{20}$$

The quantity in the () is independent of t, and so we have again related the free energy of the physical Hamiltonian near its critical temperature to that of the Hamiltonian $\bar{H}_l = \bar{H}^* + Atb^{\lambda_0 l}Q_0$ near the fixed point. So again we choose l so that $b^{\lambda_0 l}t$ is fixed, and the same results apply as before, in particular $2 - \alpha = d/\lambda_0$ and $\nu = 1/\lambda_0$. The argument is readily extended to a small value of the symmetry breaking field, leading to the same scaling result as before with the scaling variable h/t^Δ with scaling exponent $\Delta = \lambda_1/\lambda_0$.

What happens it we continue to apply $\mathcal{R}_b$? For $t < 0$ the Hamiltonian will eventually end up at the *low temperature fixed point*, and so $\bar{H}_l$ will have the same long length scale properties as this fixed point Hamiltonian, such as long range order. The scaling function will correspondingly reflect these properties. On the other hand for $t > 0$ many iterations send the Hamiltonian to the *high temperature fixed point*, and $\bar{H}_l$ in this case will reflect these properties, such as disorder at long length scales. Thus:

- The stable fixed points tell us about the qualitative properties of the phase (order, disorder etc.)

- The unstable fixed point tells us about the critical behavior near the transition temperature.