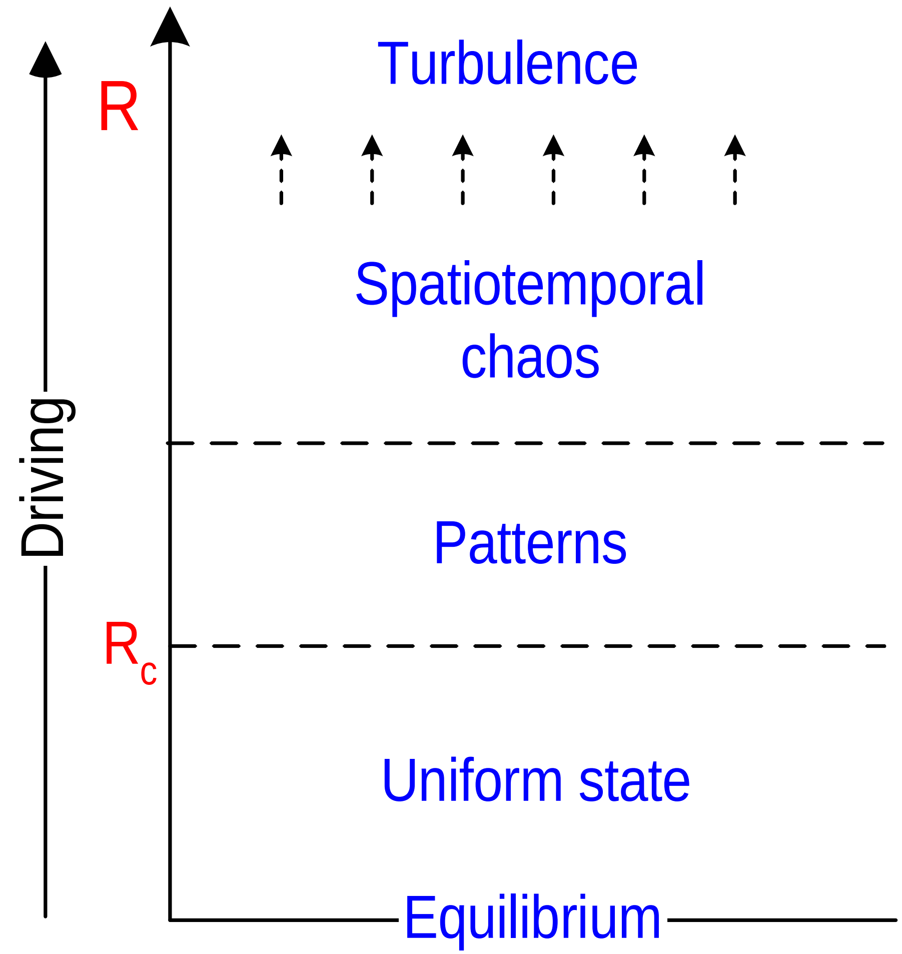
Pattern Formation in Spatially Extended Systems

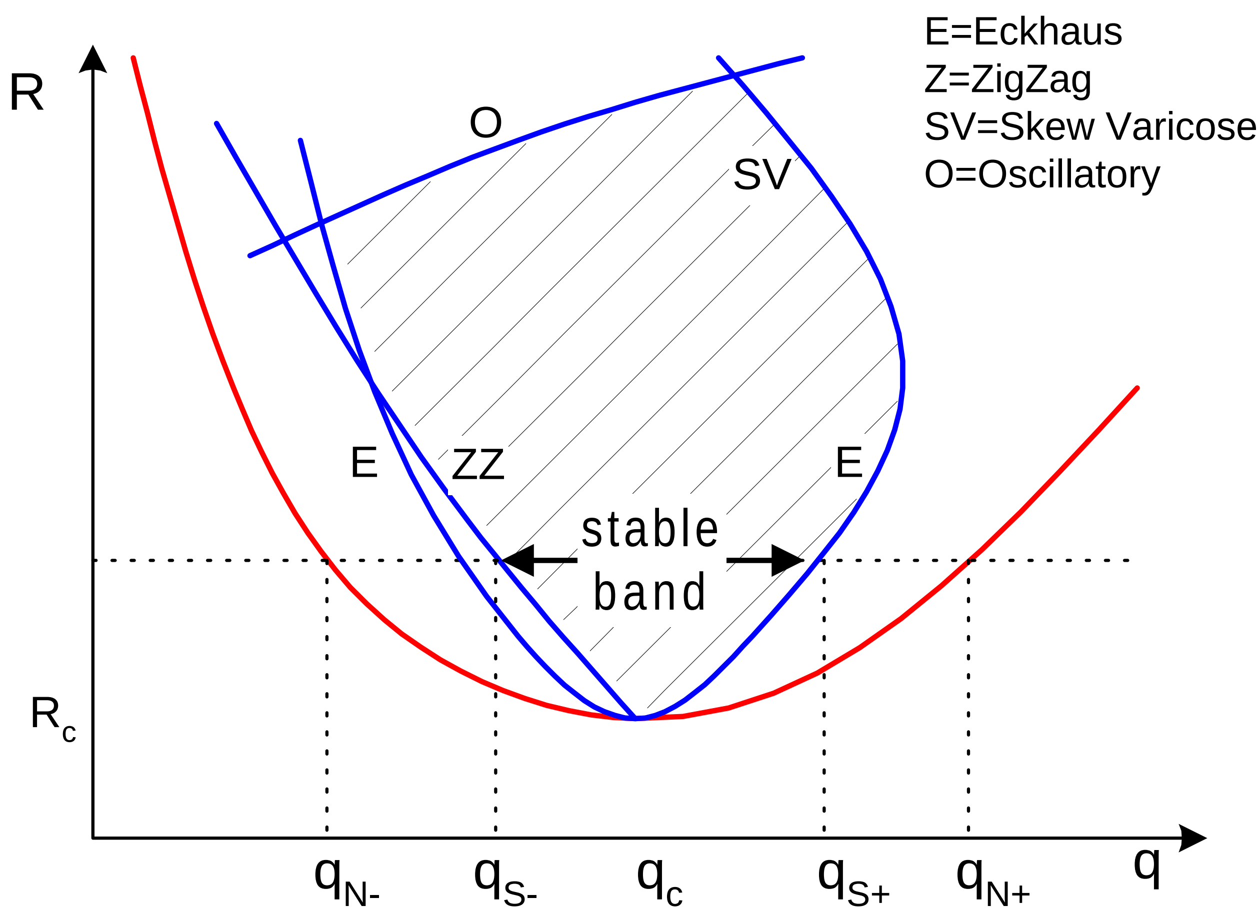
Lecture 4

Chaos

Outline

- Introduction to chaos
- Pattern chaos
- Spatiotemporal chaos
  - ◇ Definition and characterization
  - ◇ Transitions to spatiotemporal chaos and between different chaotic states
  - ◇ Coarse grained descriptions


Driving
R
Turbulence
Spatiotemporal
chaos
Patterns
R_c
Uniform state
Equilibrium


E=Eckhaus
Z=ZigZag
SV=Skew Varicose
O=Oscillatory
R
O
SV
E
ZZ
E
stable
band
R_c
q_{N-}
q_{S-}
q_c
q_{S+}
q_{N+}
q

Some Theoretical Highlights

Landau (1944) Turbulence develops by infinite sequence of transitions adding additional temporal modes and spatial complexity

Lorenz (1963) Discovered chaos in simple model of convection

Ruelle and Takens (1971) Suggested the onset of aperiodic dynamics from a low dimensional torus (quasiperiodic motion with a small number N frequencies)

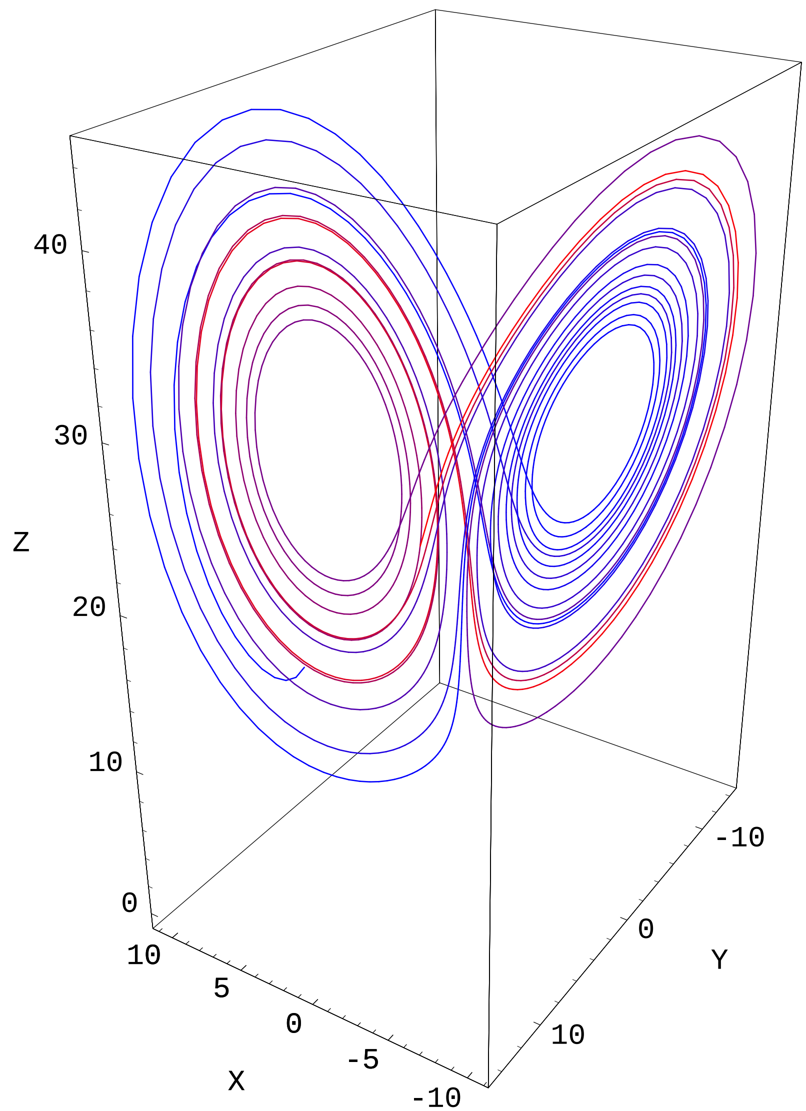
Feigenbaum (1978) Quantitative universality for period doubling route to chaos

...

Lorenz Model

$$\begin{aligned} \dot{X} &= -\sigma(X - Y) \\ \dot{Y} &= rX - Y - XZ \\ \dot{Z} &= b(XY - Z) \end{aligned}$$

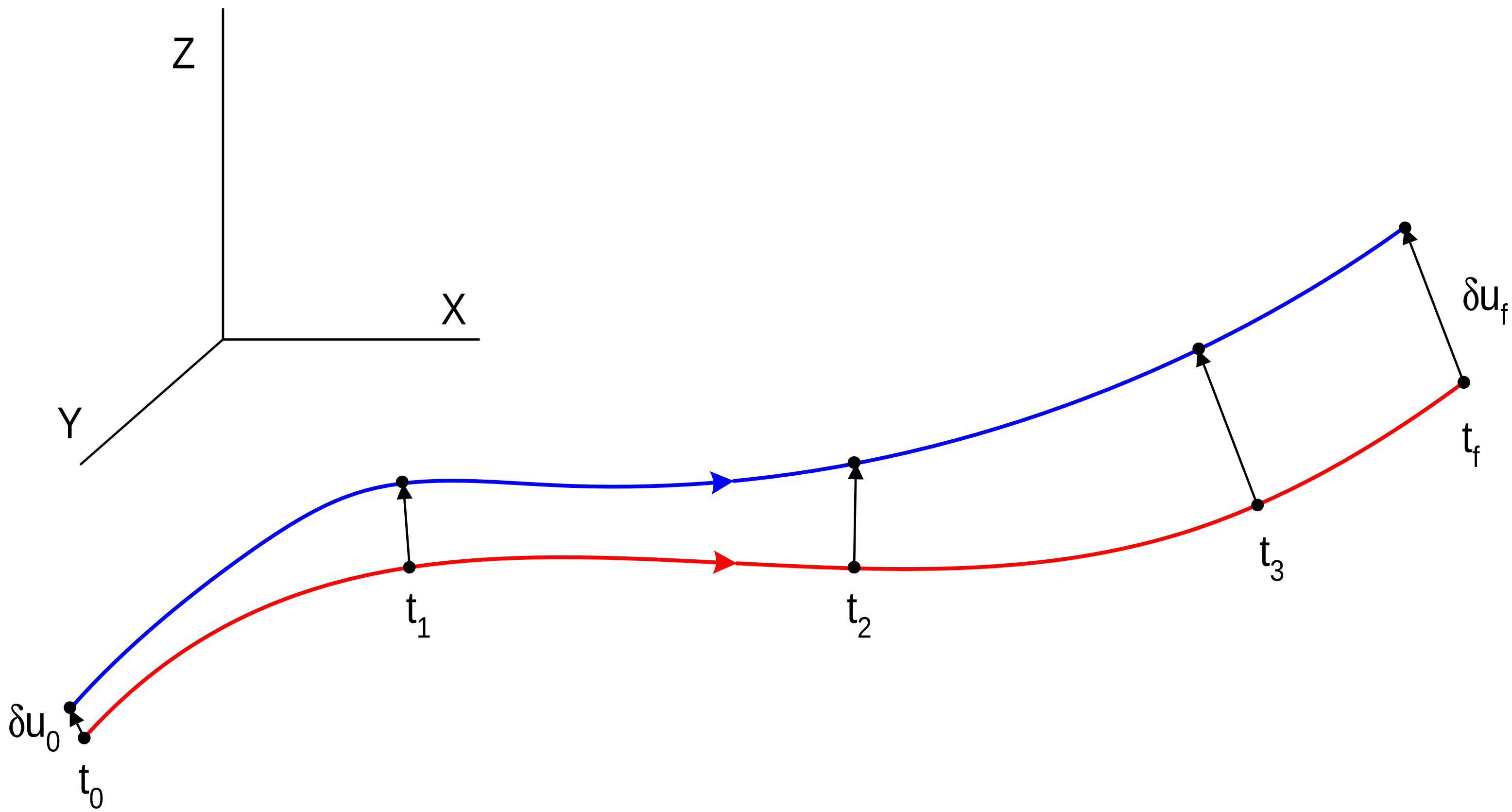
"Classic" parameter values are $b = 8/3$, $\sigma = 10$ and $r = 27$.



40
30
Z
20
10
0
10
5
0
-5
-10
X
-10
0
Y
10

Lyapunov Exponents and Eigenvectors

Quantifying the sensitive dependence on initial conditions



$$S_\lambda(t) = \ln\left|\frac{\delta\mathbf{u}_t}{\delta\mathbf{u}_0}\right|; \;\; \lambda = \lim_{t_f\to\infty}\frac{1}{t_f - t_0}S_\lambda(t_f); \qquad \delta\mathbf{u}_t \to \text{eigenvector}$$

Dimension of the Attractor

- The fractal dimension of the attractor quantifies the number of chaotic degrees of freedom.
- There are many possible definitions. Most are inaccessible to experiment and numerics for high dimensional attractors.
- I will discuss the *Lyapunov dimension* which is conjectured to be the same as the *information dimension*

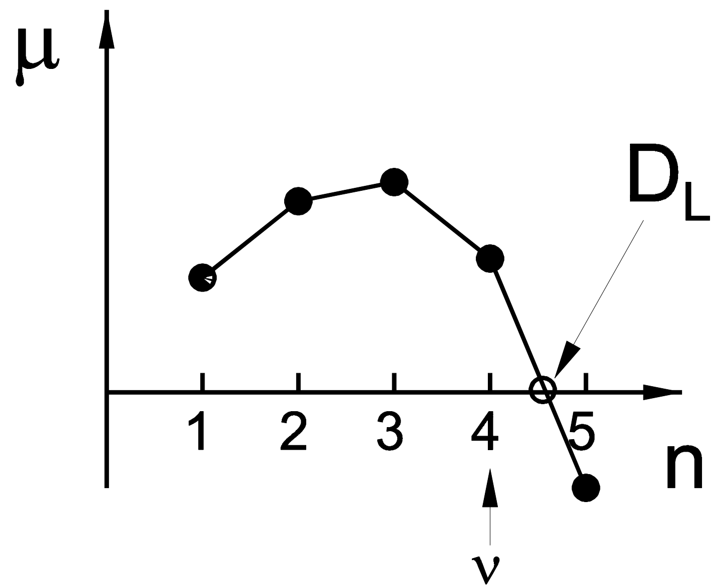
Line lengths $\rightarrow e^{\lambda_1 t}$, Areas $\rightarrow e^{(\lambda_1+\lambda_2)t}$, Volumes $\rightarrow e^{(\lambda_1+\lambda_2+\lambda_3)t}, \ldots$

Lyapunov Dimension:

$$D_L = \nu + \frac{1}{|\lambda_{\nu+1}|}\sum_{i=1}^{\nu}\lambda_i$$

where ν is the largest index such that the sum is positive.

Lyapunov dimension



Define $\mu(n) = \sum_{i=1}^{n} \lambda_i$ $\quad (\lambda_1 \geq \lambda_2 \cdots)$ with λ_i the ith Lyapunov exponent.

D_L is the interpolated value of n giving $\mu = 0$ (the dimension of the volume that neither grows nor shrinks under the evolution)

Small system chaos: some experimental highlights

Ahlers (1974) Transition from time independent flow to aperiodic flow at $R/R_c \sim 2$ (aspect ratio 5)

Gollub and Swinney (1975) Onset of aperiodic flow from time-periodic flow in Taylor-Couette

Maurer and Libchaber, Ahlers and Behringer (1978) Transition from quasiperiodic flow to aperiodic flow in small aspect ratio convection

Lichaber, Laroche, and Fauve (1982) Quantitative demonstration of the Fiegenbaum period doubling route to chaos

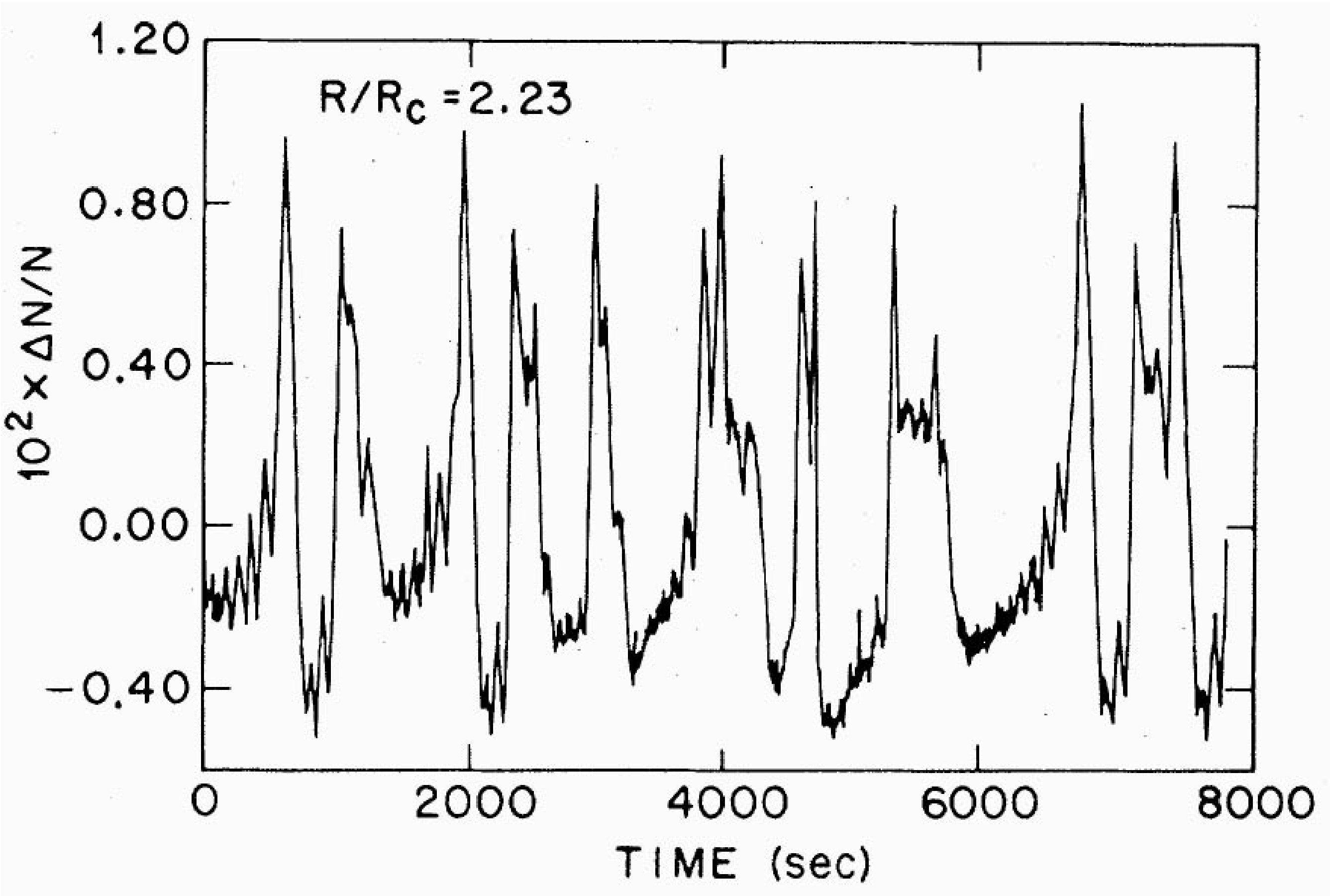
Pattern chaos

- First experiments (1974): $\Gamma = 5.27$, cryogenic (normal) liquid He^4 as fluid. High precision heat flow measurements (no flow visualization).
- Onset of aperiodic time dependence in low Reynolds number flow: relevance of chaos to "real" (continuum) systems.
- Broad power spectrum with power law decrease at large f: $P(f) \sim f^{-4}$
- Aspect ratio dependence of the onset of time dependence (1978)

Γ	2	5	57
R_t	$10R_c$	$2R_c$	$1.1R_c$

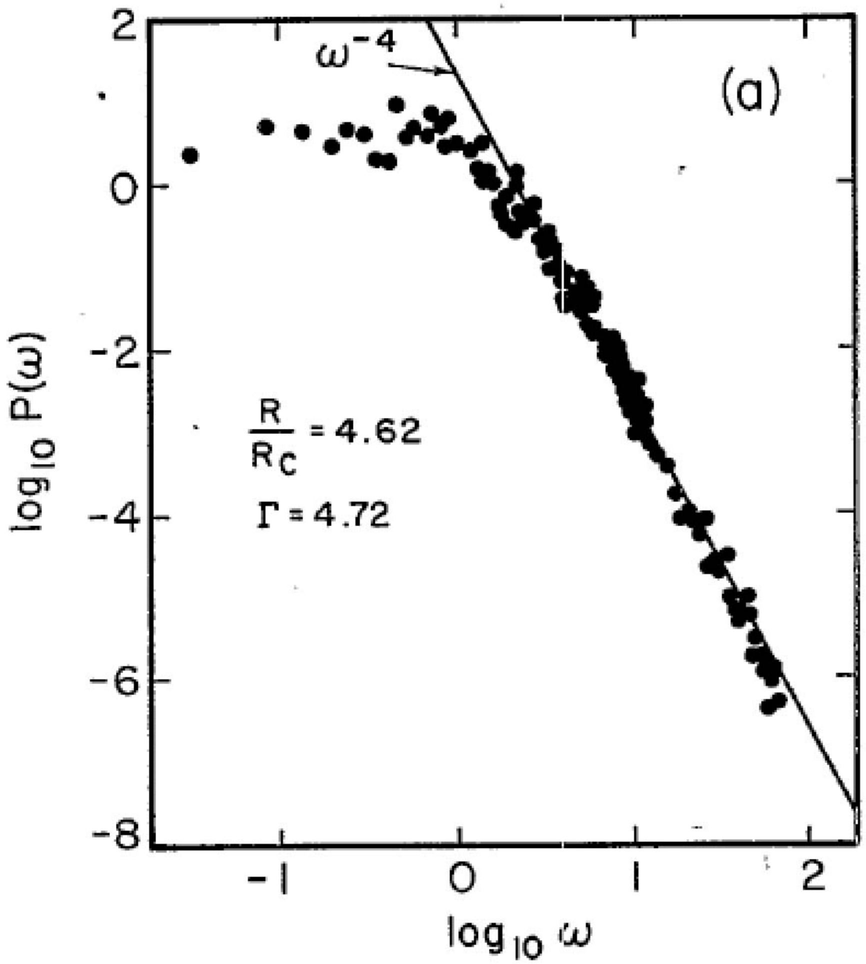
- Flow visualization (Croquette et al. 1986): $\Gamma = 7.66$, Argon
- Simulation of Generalized Swift-Hohenberg equation (Greenside, MCC, Coughran 1985)
- Simulation of full fluid equations (Paul, MCC, Fischer, and Greenside 2001)

Aperiodic time series



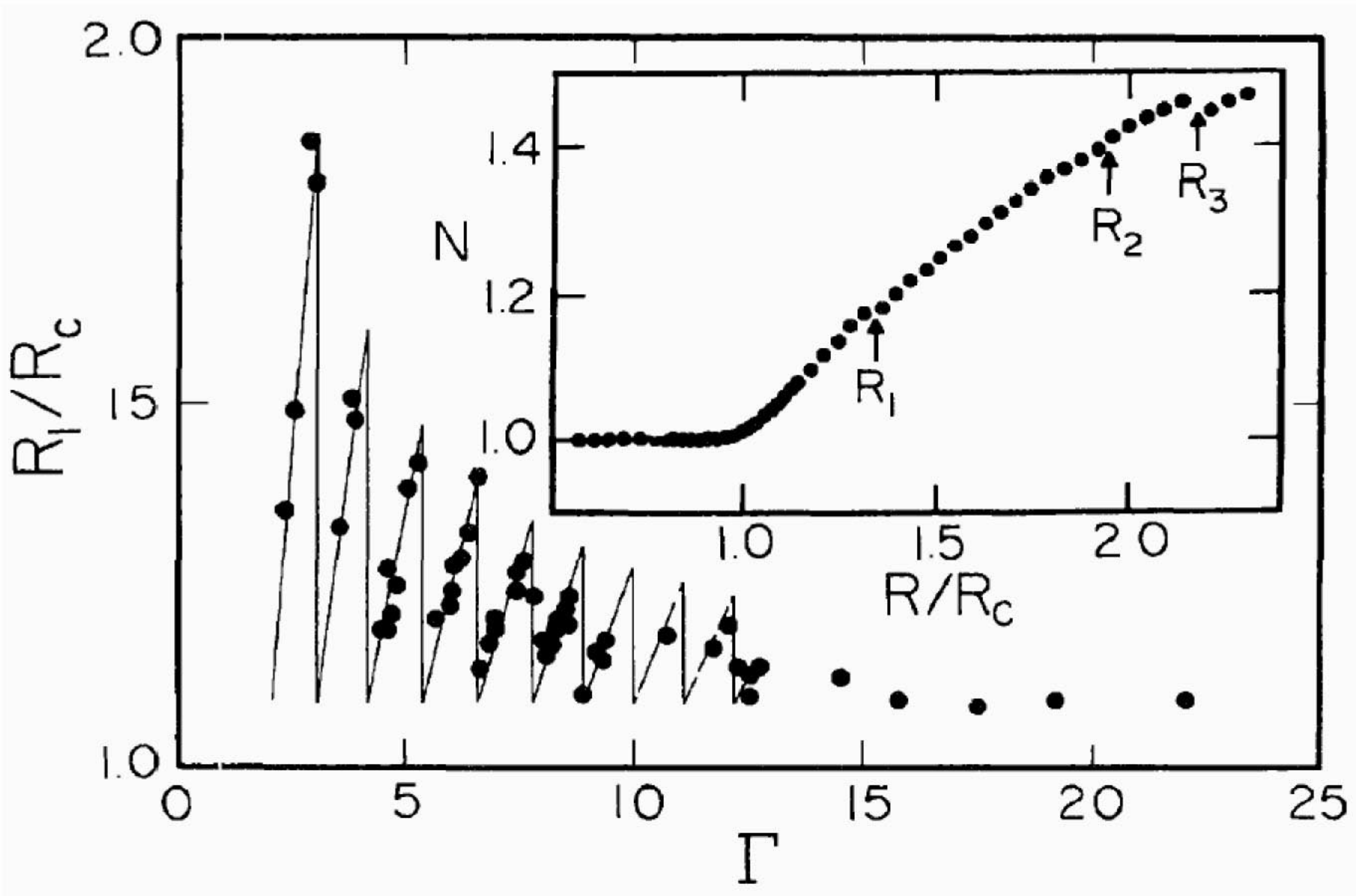
[from Ahlers 1974]

Broad power spectrum



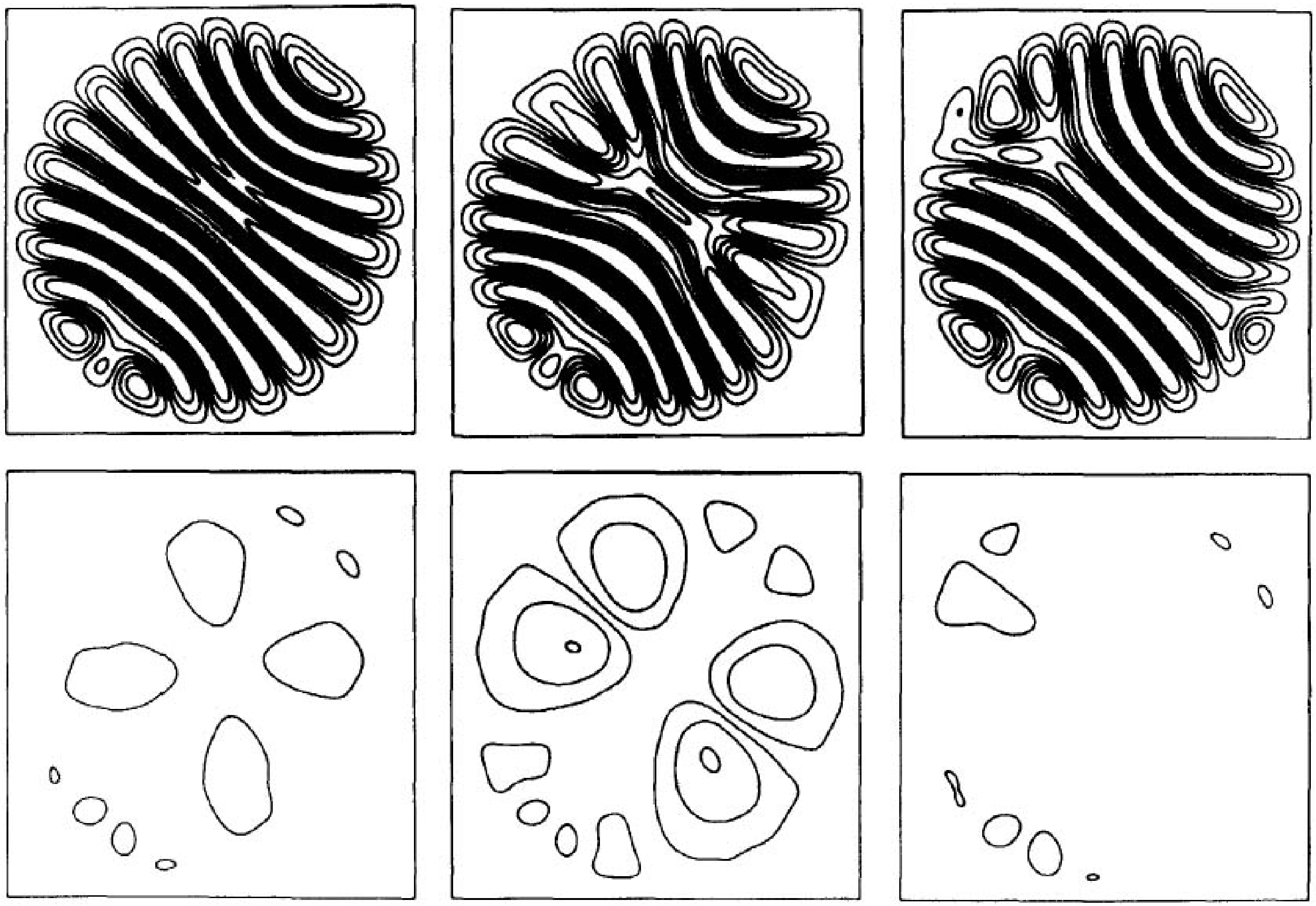
[from Ahlers and Behringer 1978]

Aspect ratio dependence



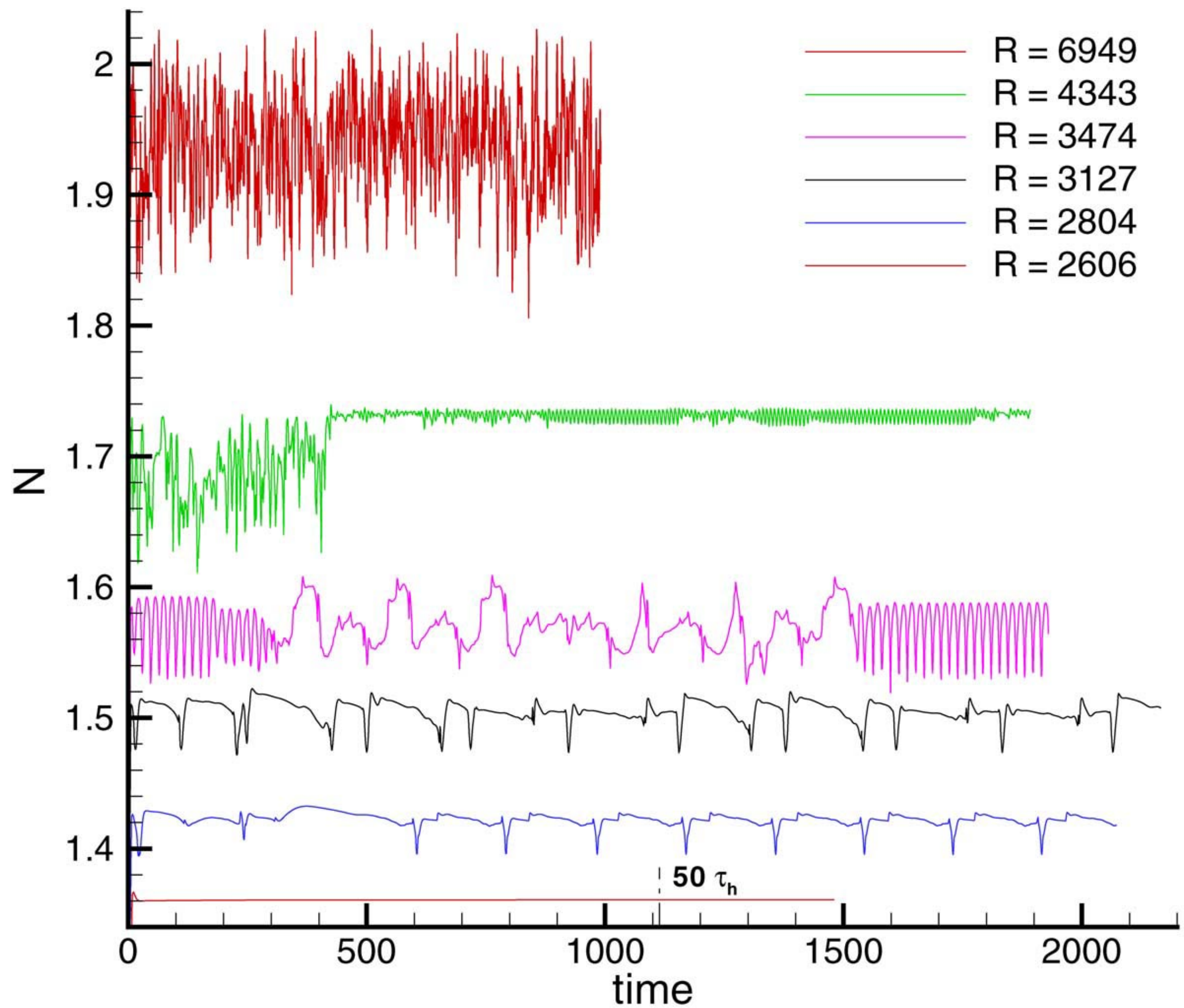
[from Gao and Behringer 1984]

Generalized Swift-Hohenberg simulations

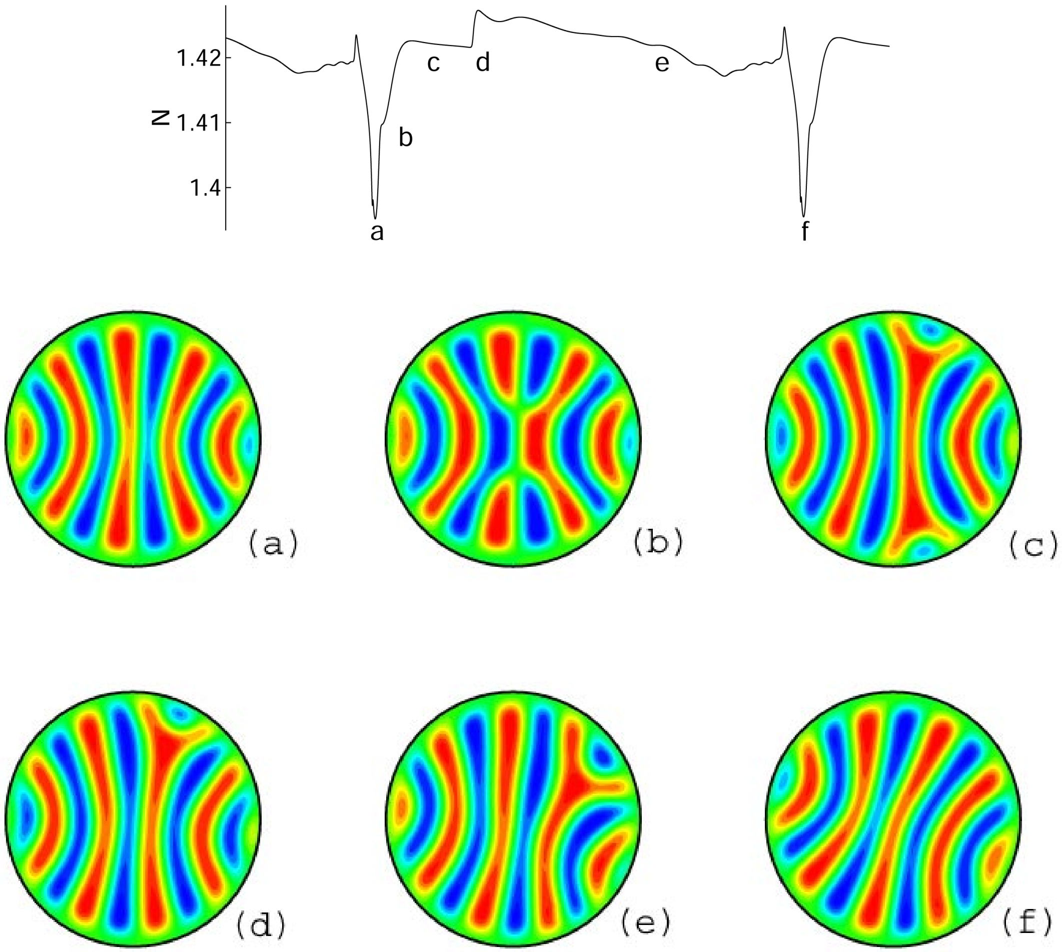


[from Greenside, MCC, Coughran 1985]

Fluid simulations

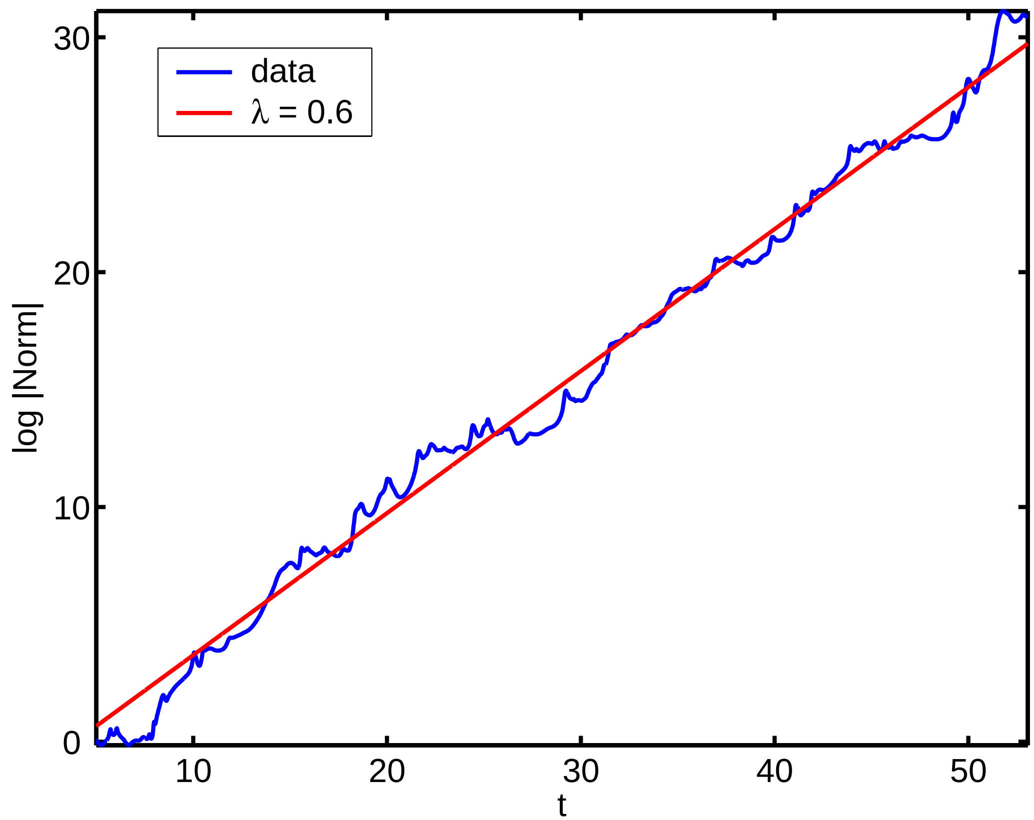


$R = 3127$ $R = 6949$



N
1.42
1.41
1.4
a
b
c
d
e
f
(a)
(b)
(c)
(d)
(e)
(f)

Lyapunov Exponent



$R = 6949$ [Scheel and MCC, 2006]

Spatiotemporal chaos

Chaos in large aspect ratio (formally $\Gamma \rightarrow \infty$) spatially homogeneous system:

- Break down of pattern to time dependent state
- Collective effect of many coupled chaotic degrees of freedom

Many natural examples of chaotic systems are closer to this idealization than to low dimensional chaos.

Systems

- Coupled Maps

$$x_{\mathbf{i}}^{(n+1)} = f(x_{\mathbf{i}}^{(n)}) + D \times \frac{1}{n} \sum_{\delta=\text{n.n.}} [f(x_{\mathbf{i}+\delta}^{(n)}) - f(x_{\mathbf{i}}^{(n)})]$$

with e.g. $f(x) = ax(1-x)$

- PDE simulations
  - ◇ Kuramoto-Sivashinsky equation

$$\partial_t u = -\partial_x^2 u - \partial_x^4 u - u\partial_x u$$

  - ◇ Amplitude equations, e.g. Complex Ginzburg-Landau Equation

$$\partial_t A = A + (1 + ic_1)\nabla^2 A - (1 - ic_3)\,|A|^2\,A$$

- Physical systems (experiment and numerics)

Challenges

- System-specific questions
- Definition and characterization
- Transitions to spatiotemporal chaos and between different chaotic states
- Coarse grained descriptions
- Control

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Ideas and methods from dynamical systems, statistical mechanics, phase transition theory …

Definition and characterization

- Narrow the phenomena
- Decide if theory, simulation, and experiment match

Characterizing spatiotemporal chaos

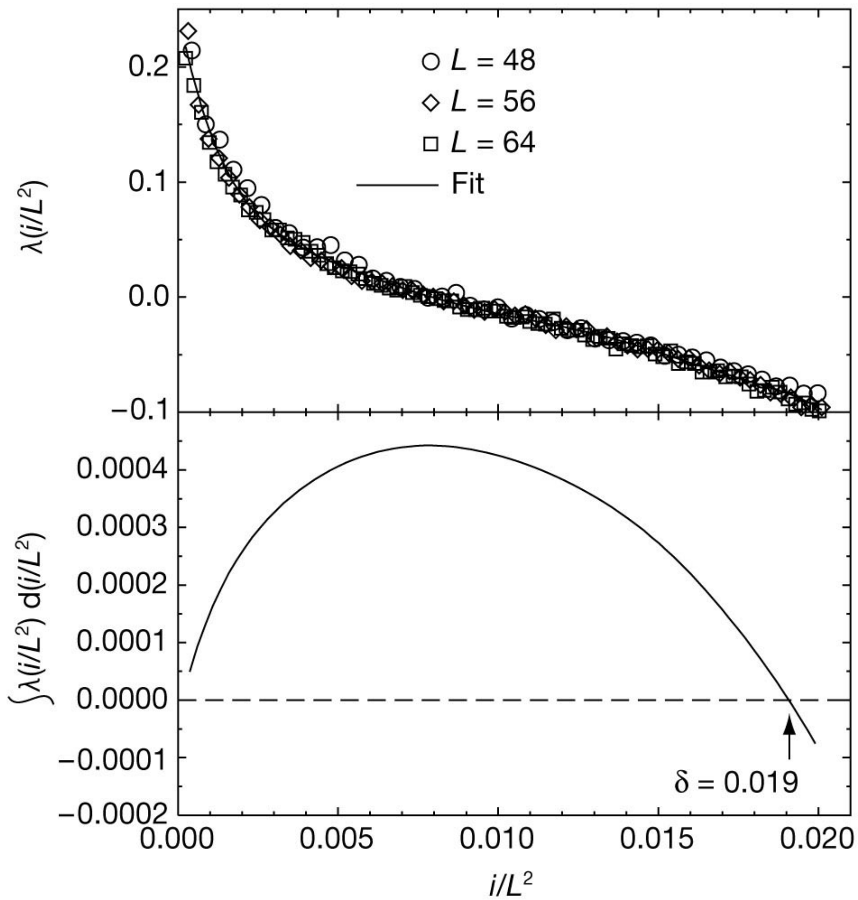
Methods from statistical physics: Correlation lengths and times, etc.

- Easy to measure, but perhaps not very insightful

Methods from dynamical systems: Lyapunov exponents and attractor dimensions.

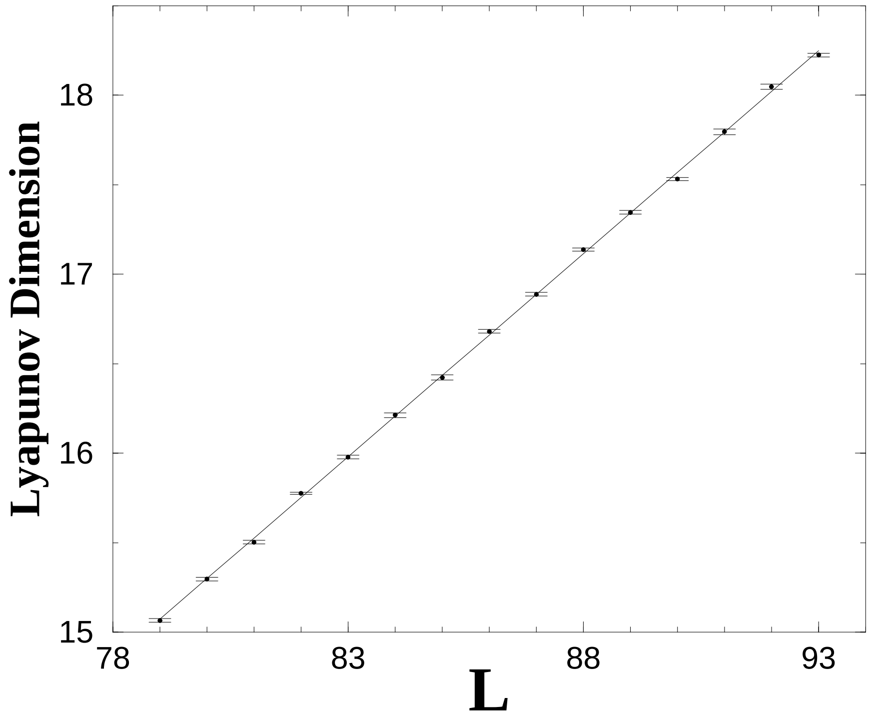
- Inaccessible in experiment, but can be measured in simulations
- Ruelle suggested that Lyapunov exponents should be intensive, and the dimension should be extensive $\propto L^d$

Lyapunov spectrum and dimension for spiral defect chaos



(Egolf et al. 2000)

Microextensivity for the 1d Kuramoto-Sivashinsky equation



(from Tajima and Greenside 2000)

Spatial aspects of sensitivity to initial conditions

Lyapunov vector for spiral defect chaos (Chiam, 2003, after Egolf et al.)

Challenges

- System-specific questions
- Definition and characterization
- Transitions to spatiotemporal chaos and between different chaotic states
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Transitions

In thermodynamic equilibrium systems the behavior may be simpler near phase transitions.

Is there universal behavior near transitions in spatiotemporal chaos (transition to STC, transitions within STC)?

If so, is the universality the same as in corresponding equilibrium systems?

Examples:

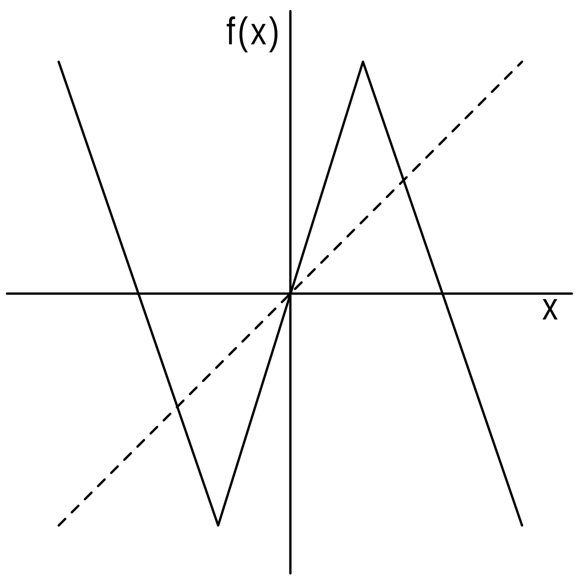
- Chaotic Ising map
- Rotating convection

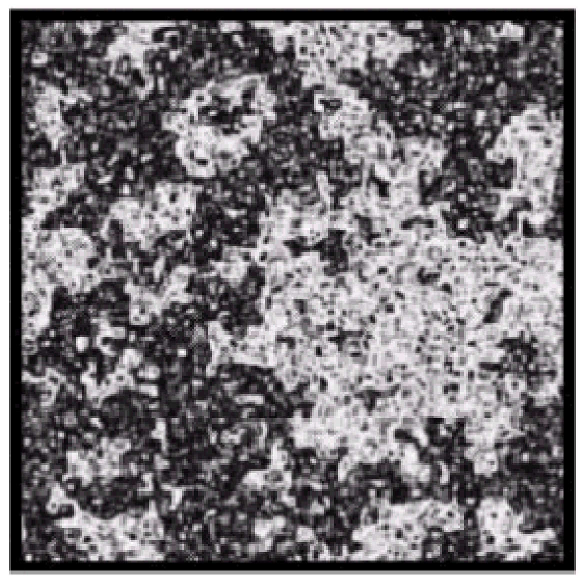
Chaotic Ising Map

J. Miller and D. Huse [Phys. Rev. **E48**, 2528 (1993)]

D. Egolf [Science **287**, 101 (2000)]

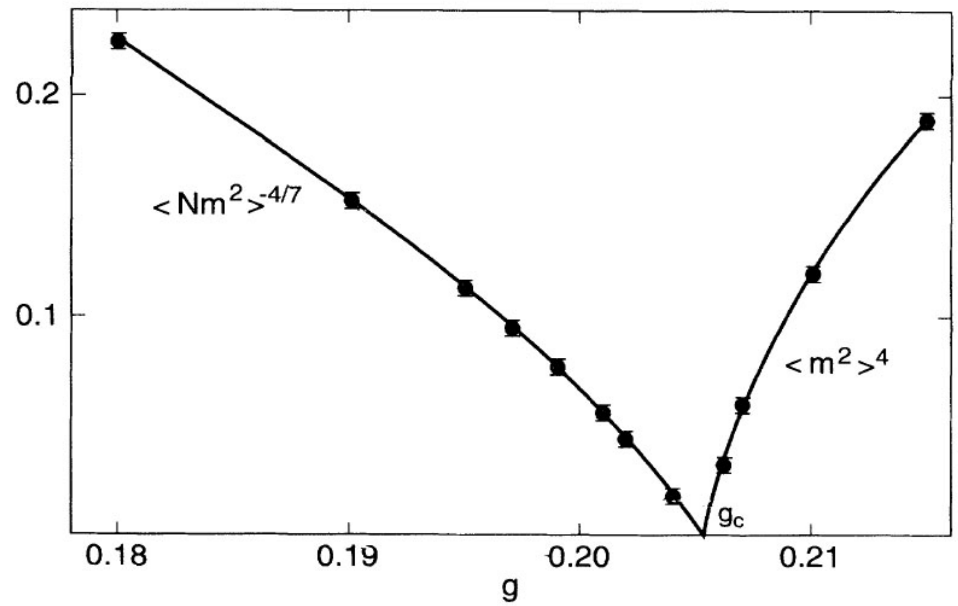
$$x_i^{(n+1)} = f(x_i^{(n)}) + g \sum_\delta (x_{i+\delta}^{(n)} - x_i^{(n)})$$





Universality

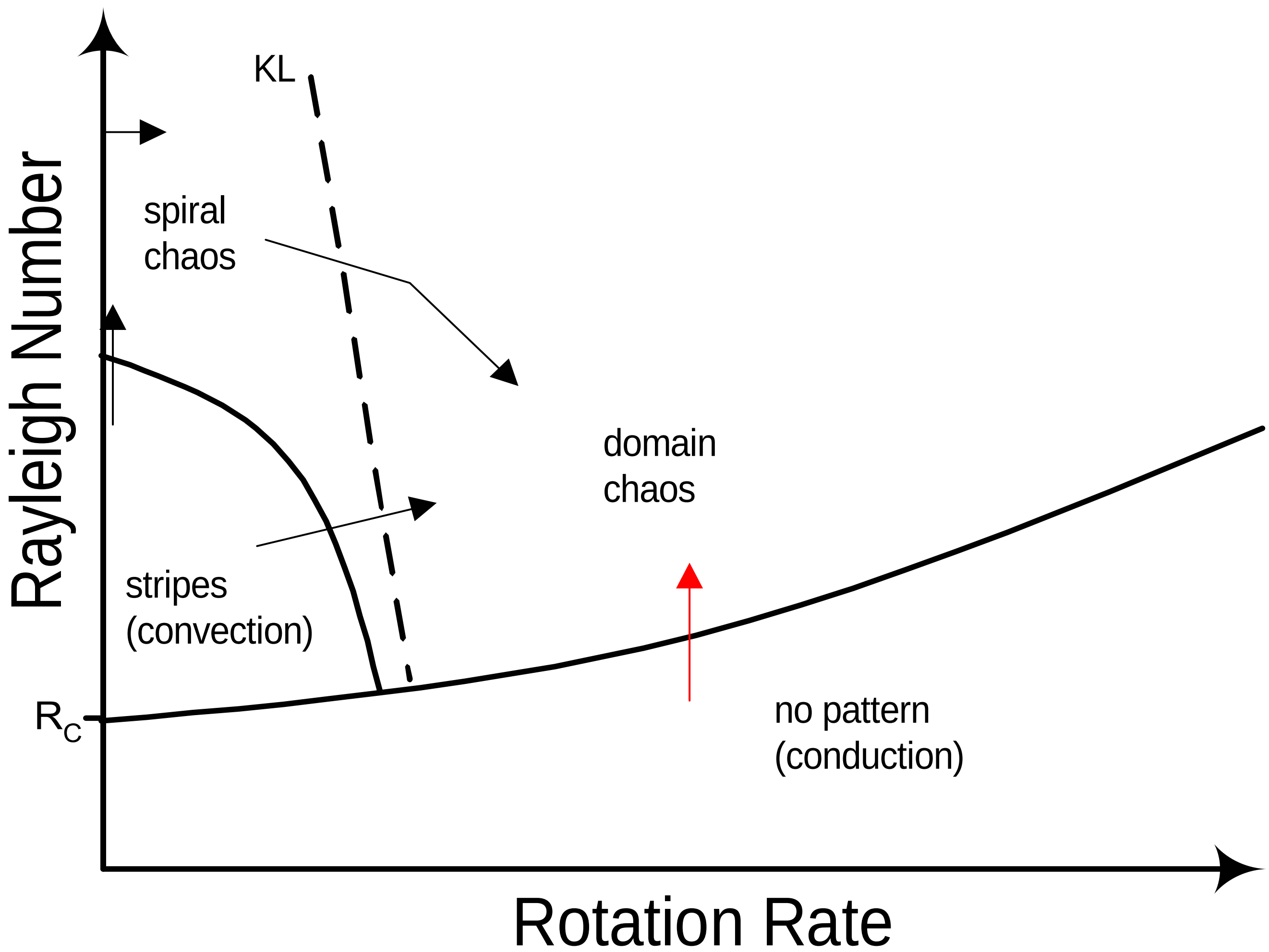
Universality apparently the same as for thermodynamic Ising system:



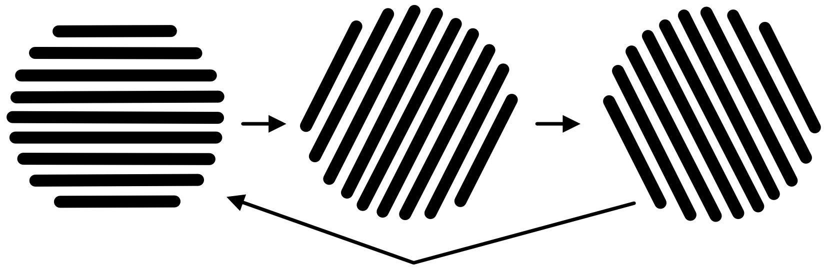
[Miller and Huse, Phys. Rev. **E48**, 2528 (1993)]

Also RNG treatment suggests non-equilibrium correction terms are *irrelevant* [Bennett and Grinstein, Phys. Rev. Lett. **55**, 657 (1985)]

Spiral and domain chaos in Rayleigh-Bénard convection



Amplitude equation description (Tu and MCC, 1992)]



Amplitudes of rolls at 3 orientations $A_i(\mathbf{r}, t),\ i = 1 \dots 3$

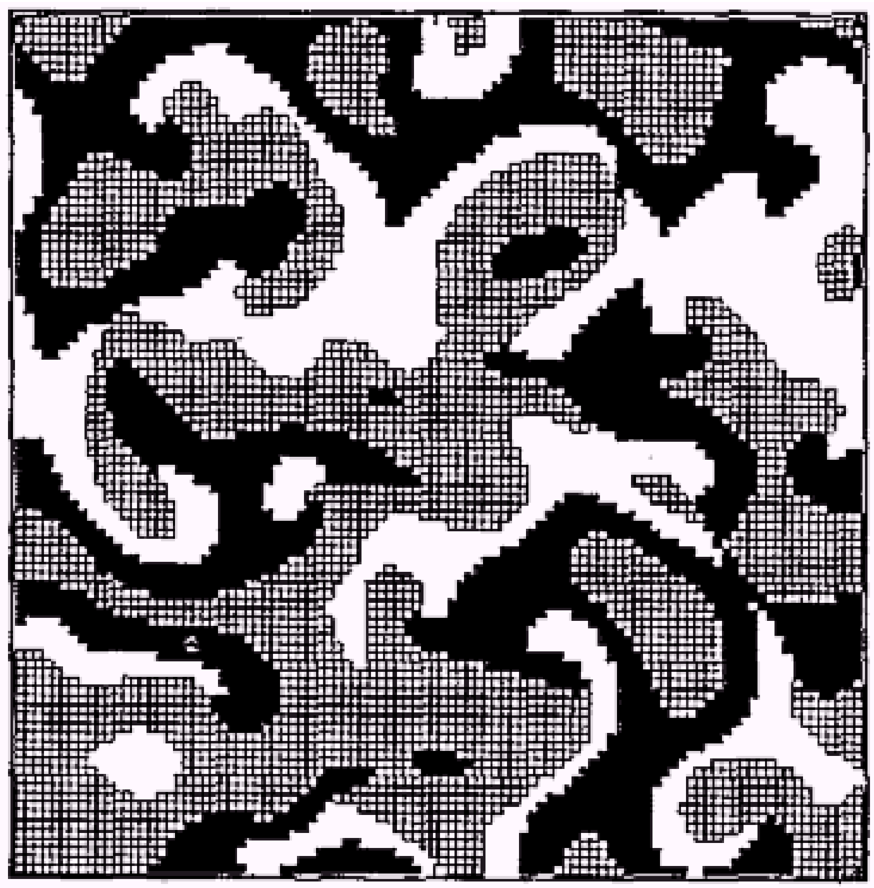
$$\partial_t A_1 = \varepsilon A_1 + \partial_{x_1}^2 A_1 - A_1(A_1^2 + g_+ A_2^2 + g_- A_3^2)$$
$$\partial_t A_2 = \varepsilon A_2 + \partial_{x_2}^2 A_2 - A_2(A_2^2 + g_+ A_3^2 + g_- A_1^2)$$
$$\partial_t A_3 = \varepsilon A_3 + \partial_{x_3}^2 A_3 - A_3(A_3^2 + g_+ A_1^2 + g_- A_2^2)$$

where $\varepsilon = (R - R_c(\Omega))/R_c(\Omega)$

Length scale $\xi \sim \varepsilon^{-1/2}$

Time scale $\tau \sim \varepsilon^{-1}$

Chaotic dynamics



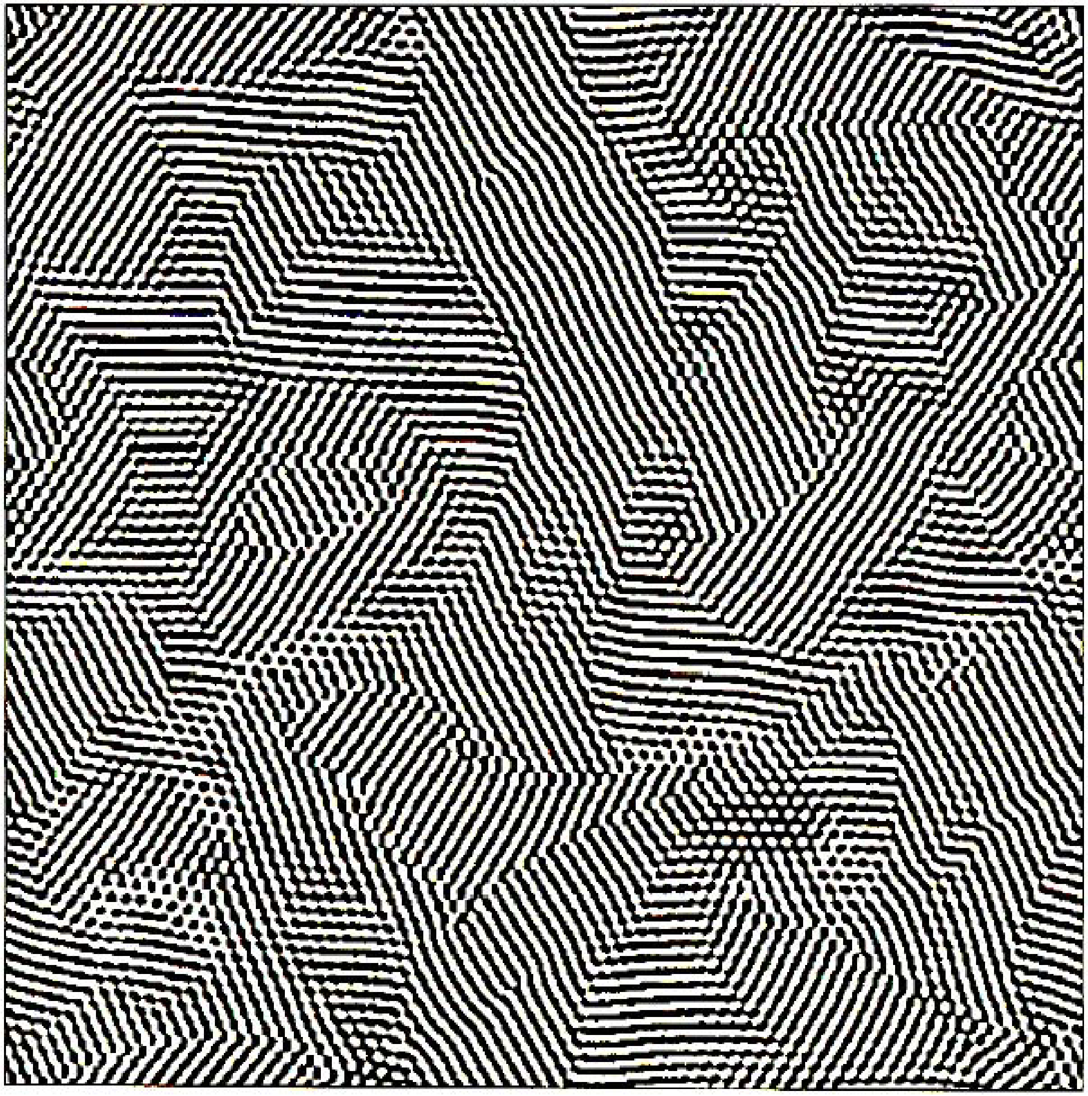
Suggests characteristic lengths and times scaling as $\xi \sim \varepsilon^{-1/2}$ and $\tau \sim \varepsilon^{-1}$

Generalized Swift-Hohenberg simulations of domain chaos

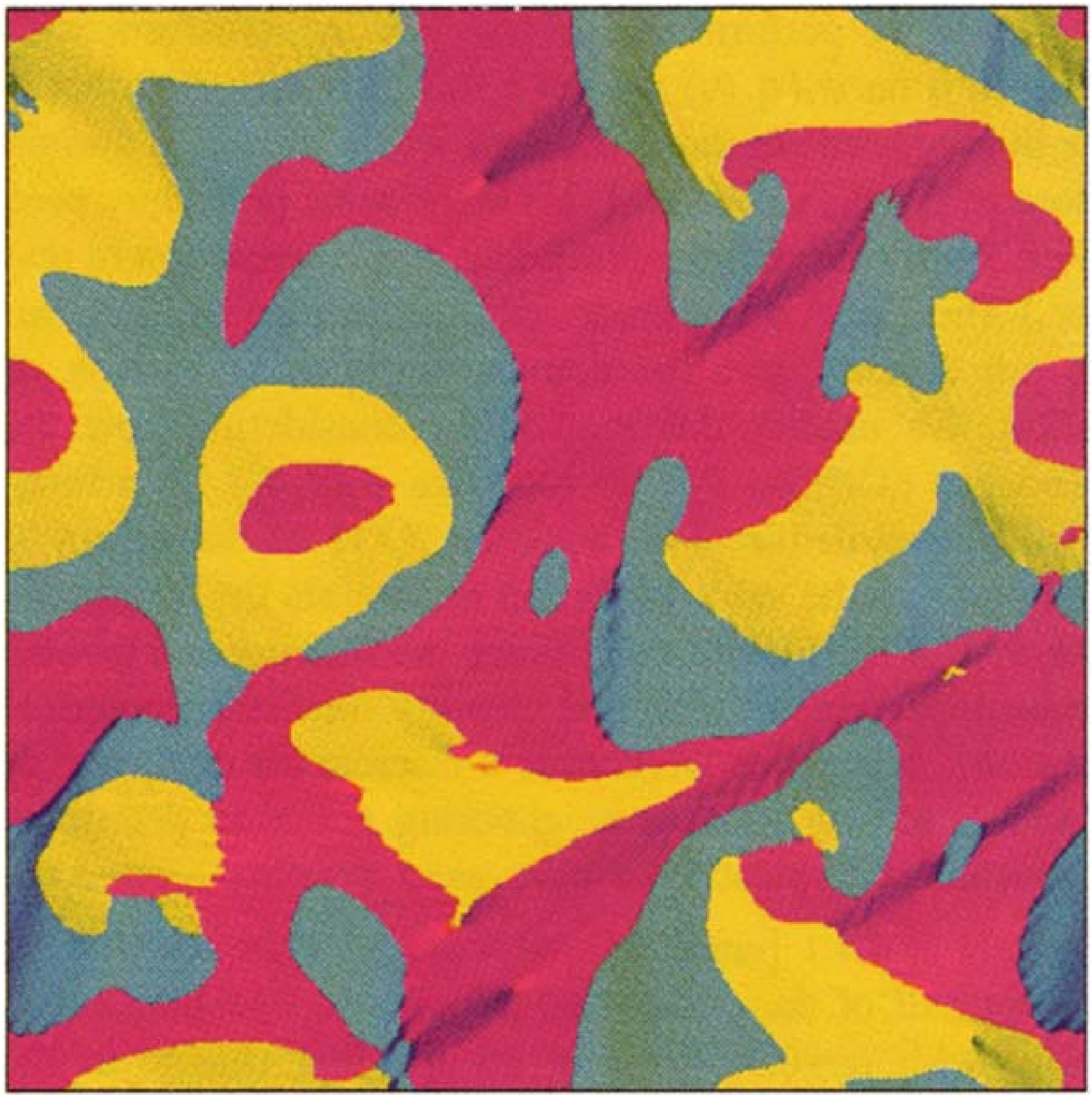
$$\frac{\partial \psi}{\partial t} = \varepsilon \psi + (\nabla^2 + 1)^2 \psi - g_1 \psi^3$$
$$+ g_2 \hat{\mathbf{z}} \cdot \nabla \times [(\nabla \psi)^2 \nabla \psi] + g_3 \nabla \cdot [(\nabla \psi)^2 \nabla \psi]$$

e.g. $g_1 = 1$, $g_2 = 2.6$, $g_3 = 1, 5$, $\varepsilon = 0.1$ (MCC, Meiron and Tu, 1994)

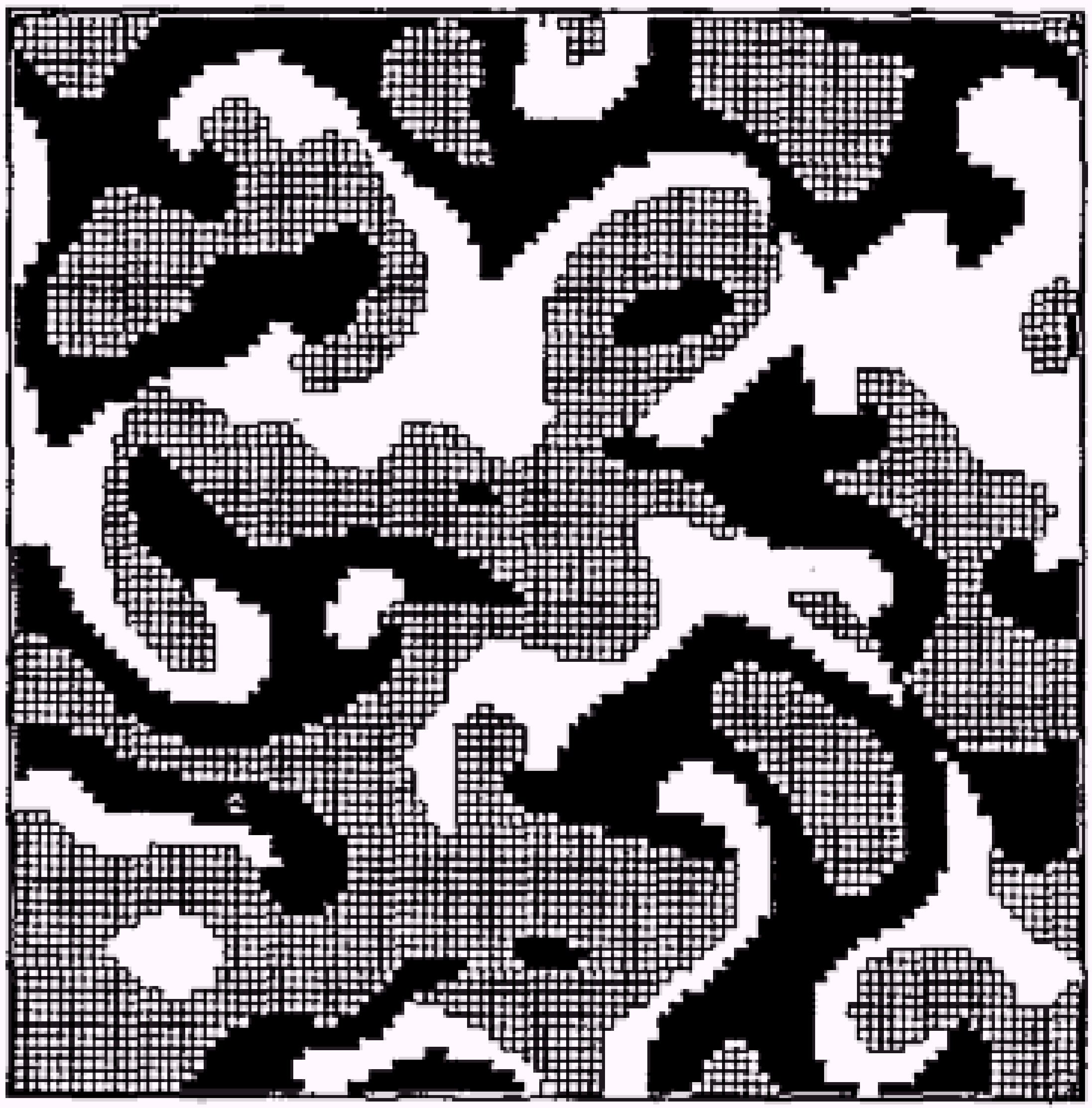
GSH stripes



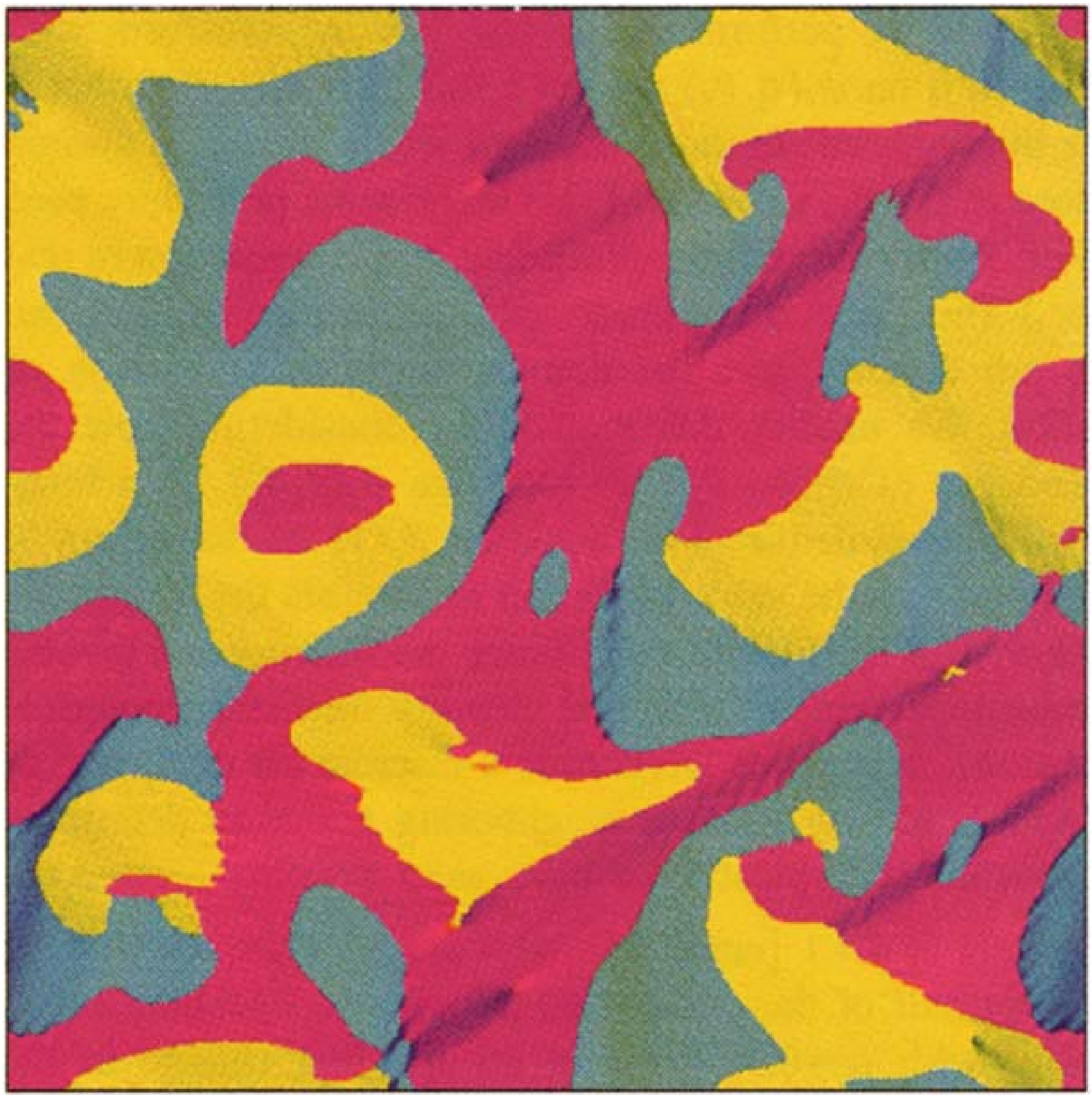
GSH orientations



cf. amplitude simulations



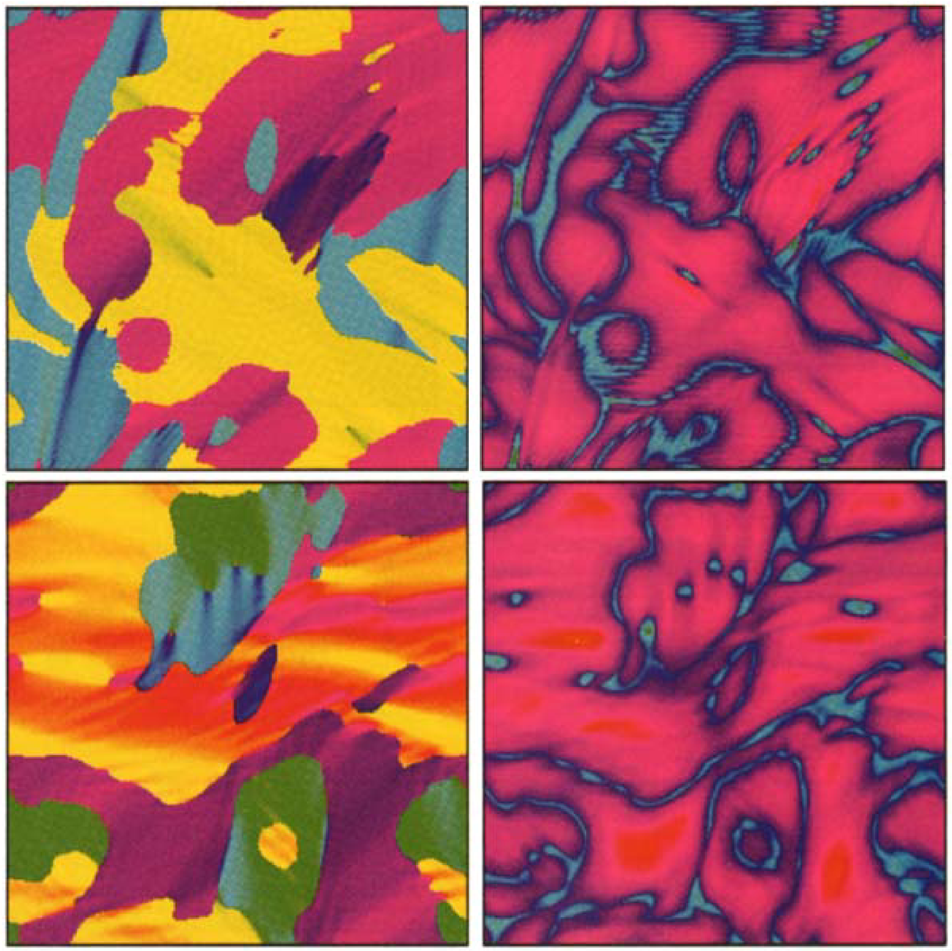
GSH orientations



GSH domain walls

Visual demonstration of scaling

$\varepsilon = 0.03$, $\Gamma = 180\pi$



$\varepsilon = 0.2$, $\Gamma = 62\pi$

Challenges

- System-specific questions
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- Transitions to spatiotemporal chaos and between different chaotic states
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- Control

Coarse grained description

- Can we find simplified descriptions of spatiotemporal chaotic systems at large length scales?
    - ◇ conserved quantity (cf. hydrodynamics)
    - ◇ near continuous transition
    - ◇ collective motion such as defects
- Is the simplified description analogous to a thermodynamic equilibrium system?

Rough argument

Rough argument

Expect a Langevin description at large scales

$$\partial_t \mathbf{y} = \mathbf{D}(\mathbf{y}) + \eta$$

y is vector of large length scale variables, **D** is some effective deterministic dynamics, and η is noise coming from small scale chaotic dynamics.

Rough argument

Expect a Langevin description at large scales

$$\partial_t \mathbf{y} = \mathbf{D}(\mathbf{y}) + \eta$$

$\mathbf{y}$ is vector of large length scale variables, $\mathbf{D}$ is some effective deterministic dynamics, and η is noise coming from small scale chaotic dynamics.

Since η represents the effect of many small scale fast chaotic degrees of freedom acting on the large scales we might expect it to be Gaussian and white

$$\left\langle \eta_i(\mathbf{r}, t)\eta_j(\mathbf{r}', t') \right\rangle = \Omega_{ij}\delta(\mathbf{r} - \mathbf{r}')\delta(t - t')$$

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In systems deriving from a microscopic Hamiltonian dynamics *constraints* relate the noise Ω_{ij} and the deterministic terms $\mathbf{D}$ (the fluctuation-dissipation theorem).

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In systems deriving from a microscopic Hamiltonian dynamics *constraints* relate the noise Ω_{ij} and the deterministic terms $\mathbf{D}$ (the fluctuation-dissipation theorem).

In systems based on a dissipative small scale dynamics, *if* the dominant macroscopic dynamics is sufficiently simple, or sufficiently constrained by symmetries, these relationships may *happen* to occur.

Examples

- Chaotic Kuramoto-Sivashinsky dynamics reduces to noisy Burgers equation [Zaleski (1989)]
- Chaotic Ising map model near the transition
    - ◇ Langevin equation for dynamics of domain walls same as in equilibrium system [Miller and Huse, (1993)]
    - ◇ Coarse grained configurations satisfy detailed balance and have a distribution given by an effective free energy [Egolf, Science **287**, 101 (2000)]
- Defect dynamics description of 2D Complex Ginzburg Landau chaos [Brito et al., Phys. Rev. Lett. **90**, 063801 (2003)]

Conclusions

In this lecture I introduced some of the basic ideas of chaos, and discussed the application of these ideas to pattern forming systems.

I discussed one of the first experiments on chaos in continuum systems where the chaotic dynamics involves many of the ideas discussed in the previous lectures.

I then introduced spatiotemporal chaos, which remains a poorly characterized and understood phenomenon, and discussed

- Definition and characterization
- Transitions
- Coarse grained descriptions