Pattern Formation in Spatially Extended Systems

Lecture 1

- some pictures
- linear instability
- nonlinear saturation
- stability balloon
- amplitude equation

Pattern formation

The formation of spatial structure in systems that are:

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Pattern formation

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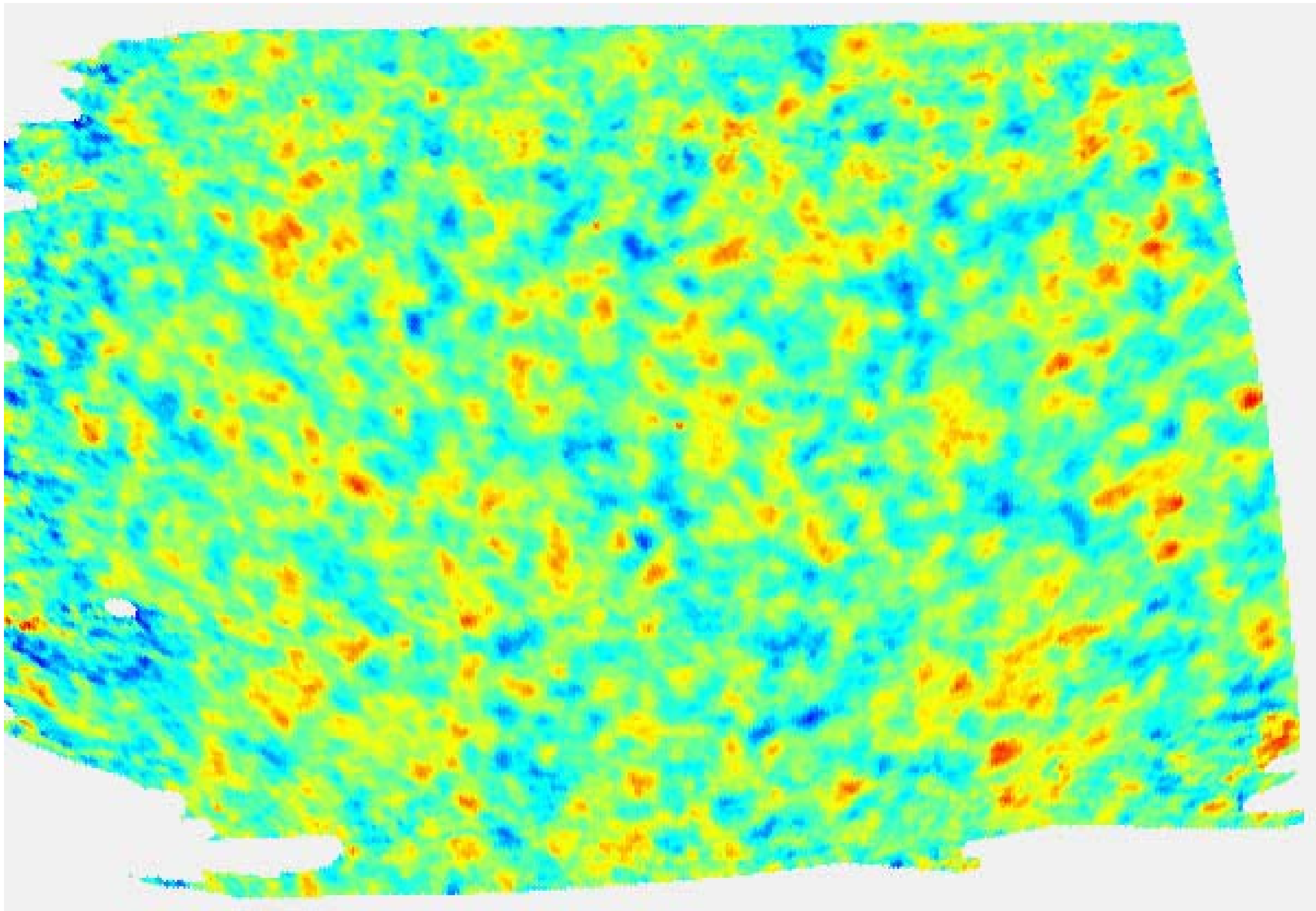
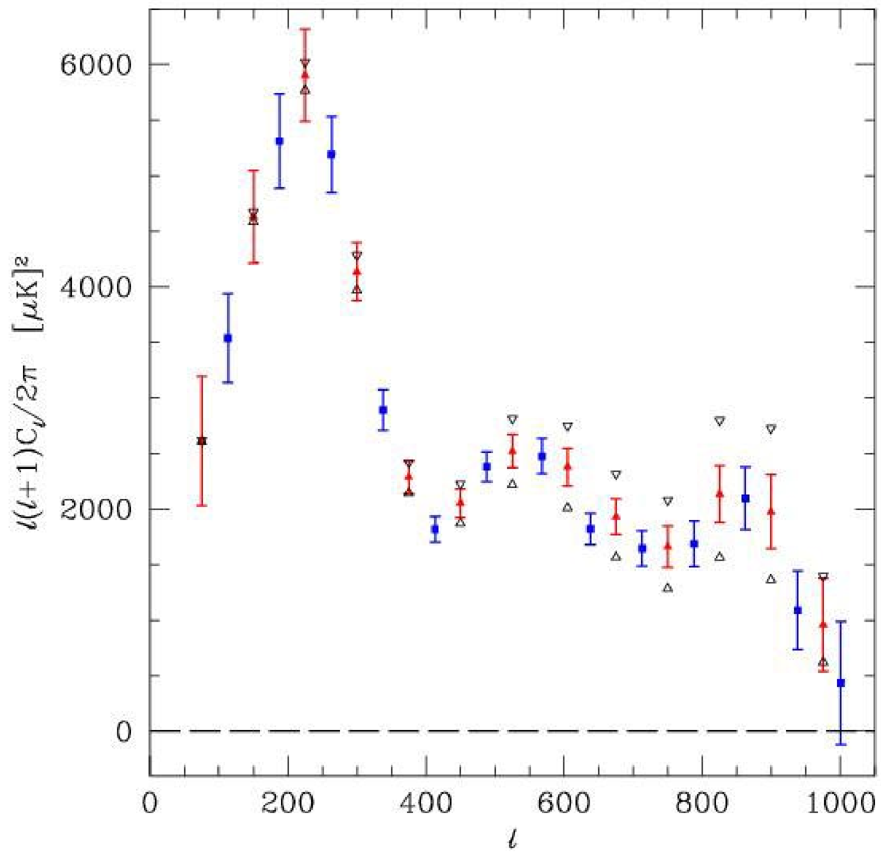
- driven…
- dissipative…
- therefore, nonequilibrium…
- characterized by energy injection, transport, and dissipation…
- and so cannot be described in terms of the minimization of a (free) energy

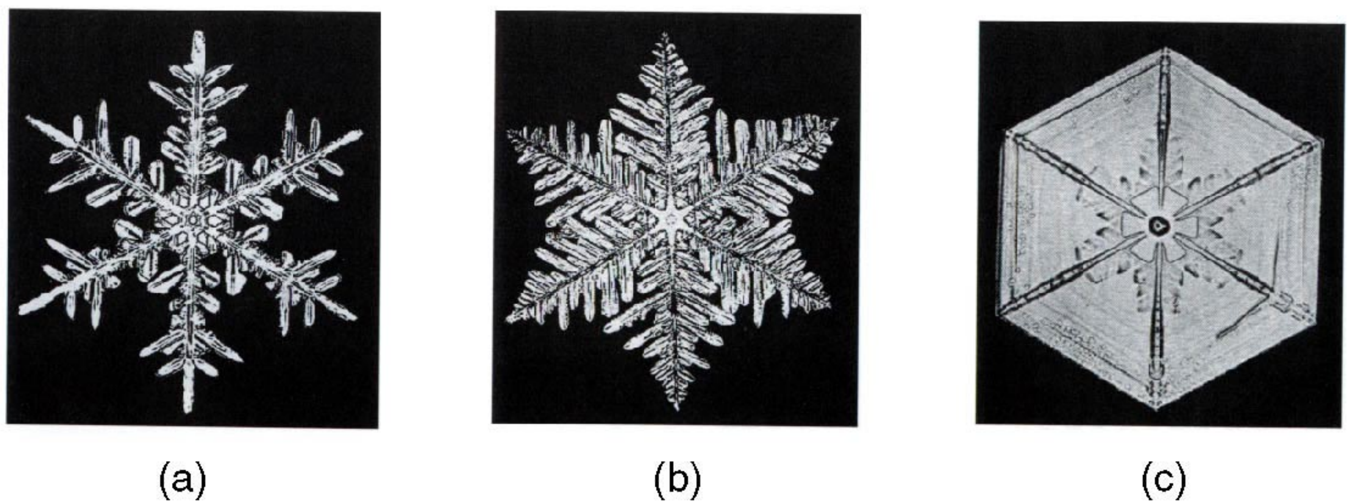


Microwave background from a 30° by 100° portion of the sky showing fluctuations of about $10^{-4}K$. (Boomerang project).

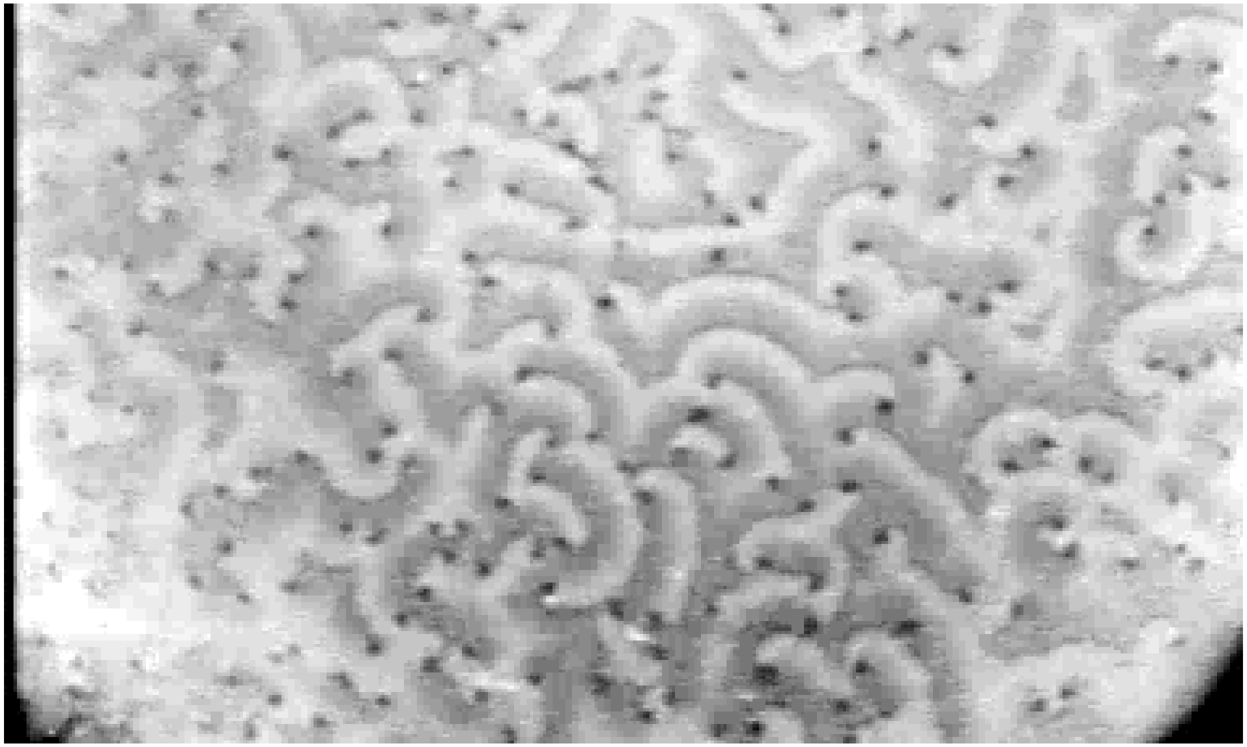


Angular power spectrum of Boomerang data.

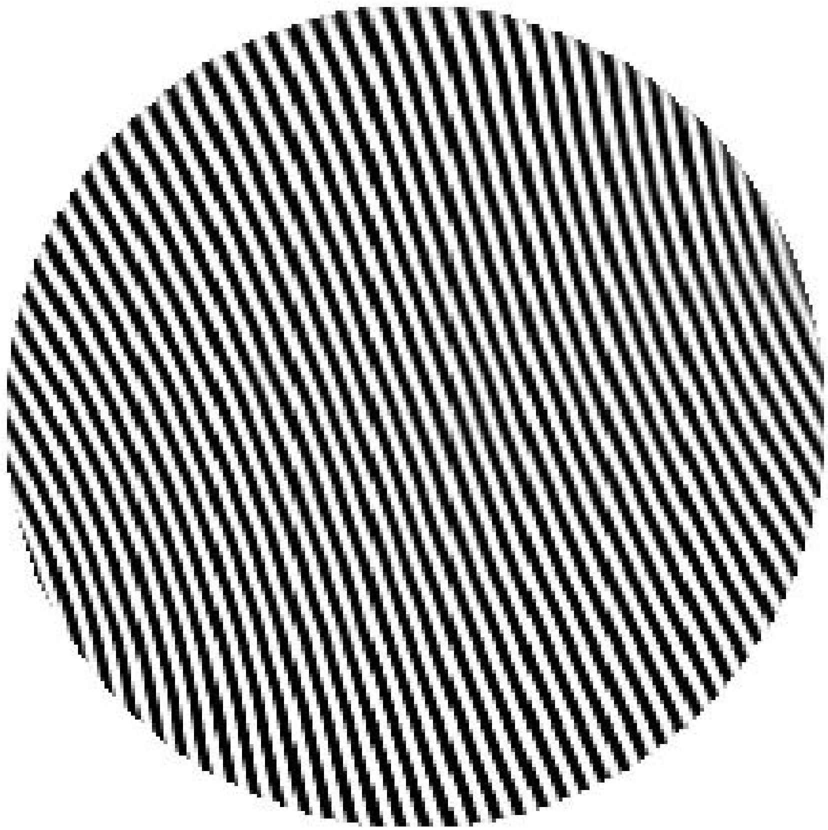
Wind-swept sand at the Kelso dune field of the Mojave desert in California. The ripple spacing is about 10 cm.



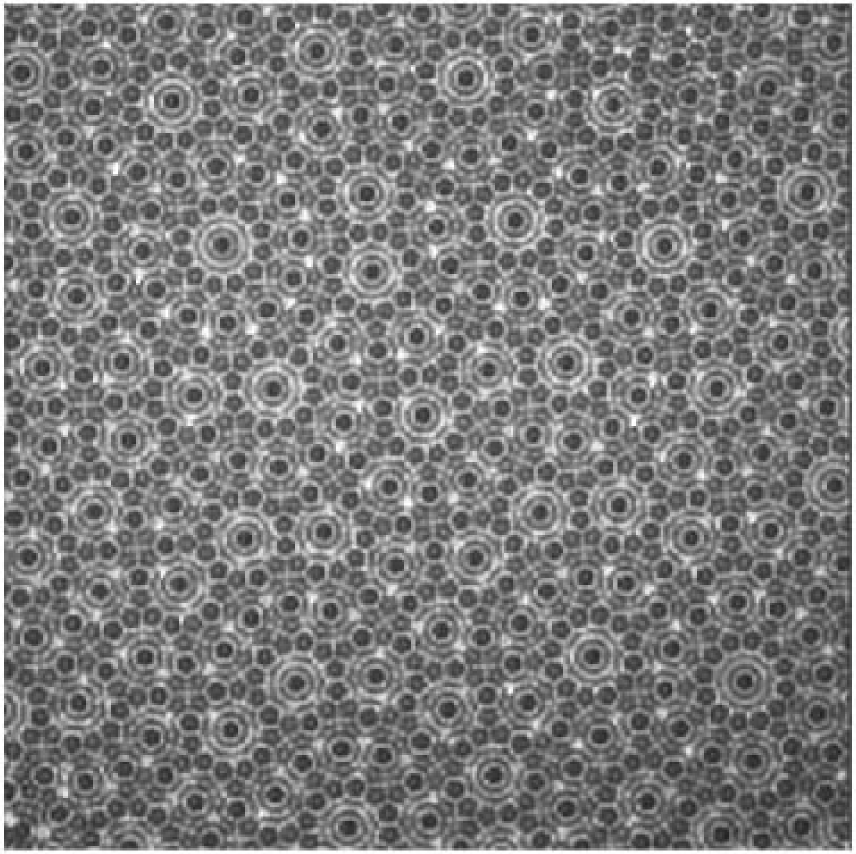
Photomicrographs of snowflakes by Wilson Bentley.



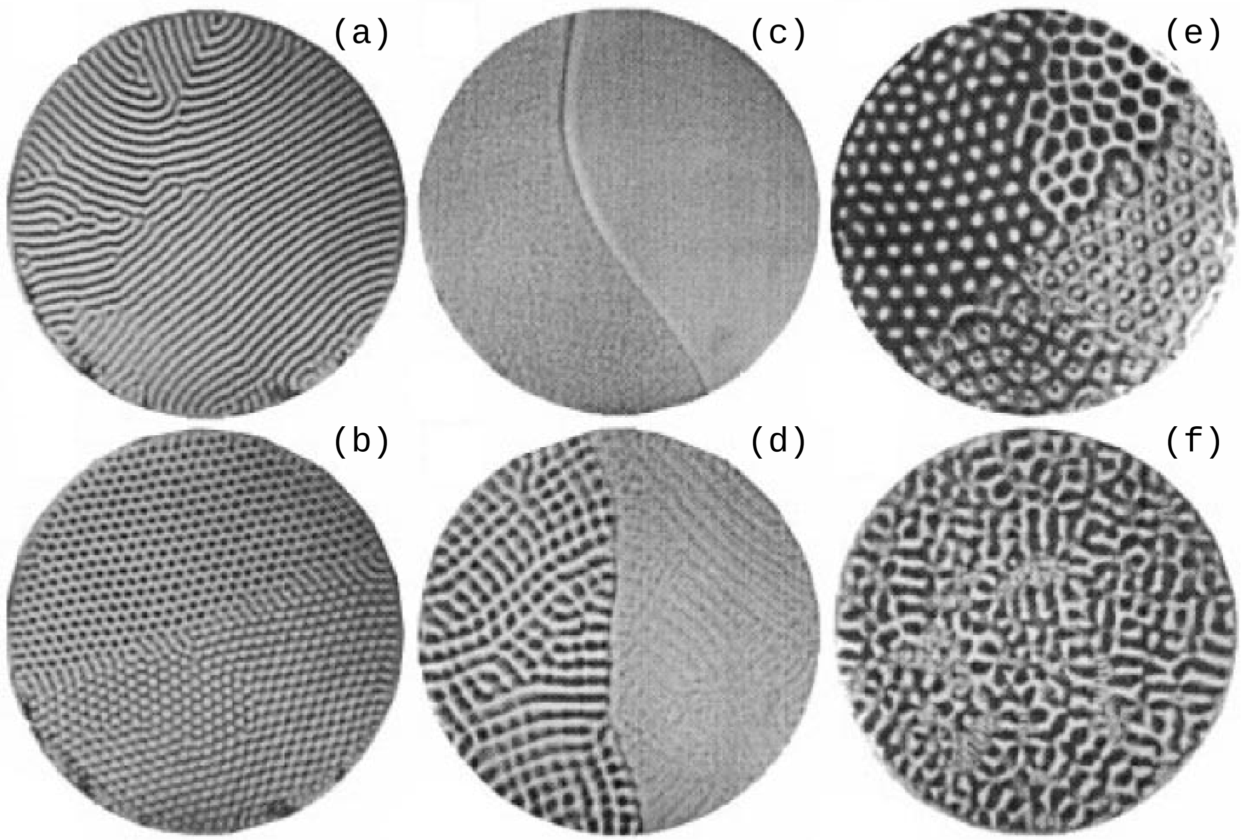
Starving slime mold colony in the early stages of aggregation. The light regions correspond to cells that are moving with a speed of about $400\,\mu\,\mathrm{min}^{-1}$ towards higher secretant concentrations (chemotaxis). [Figure from Florian Siegert].



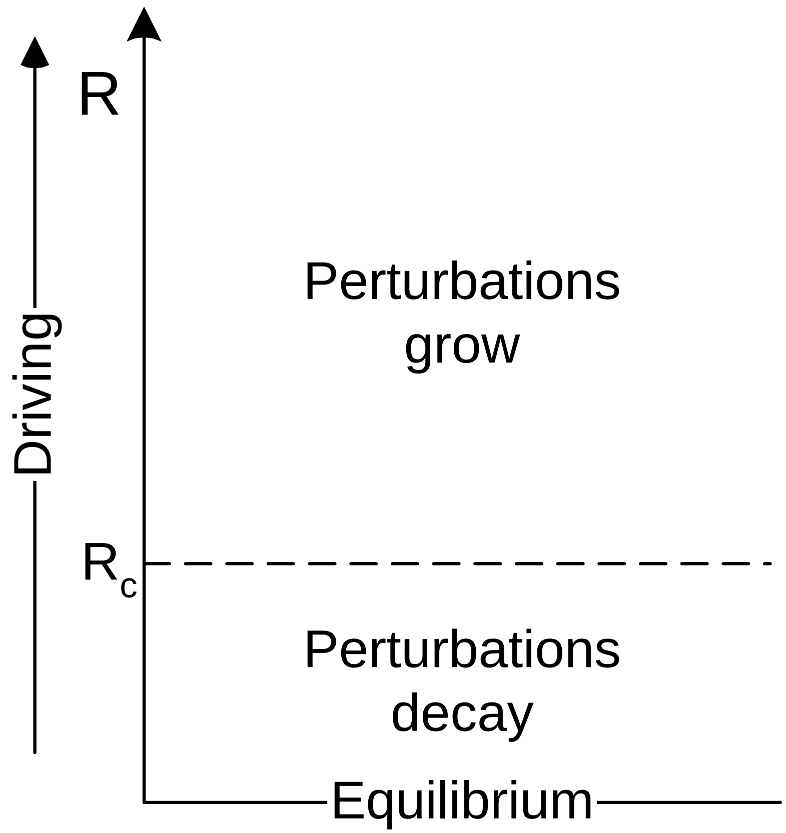
From the website of G. Ahlers

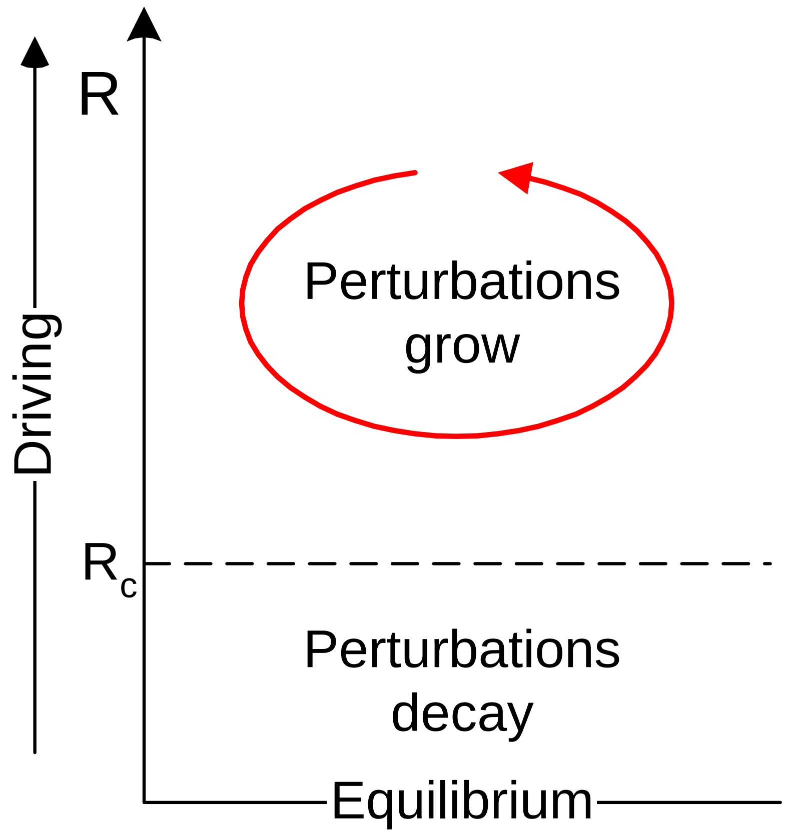


Vertically shaken layer of fluid [From Kudrolli, Pier, and Gollub]

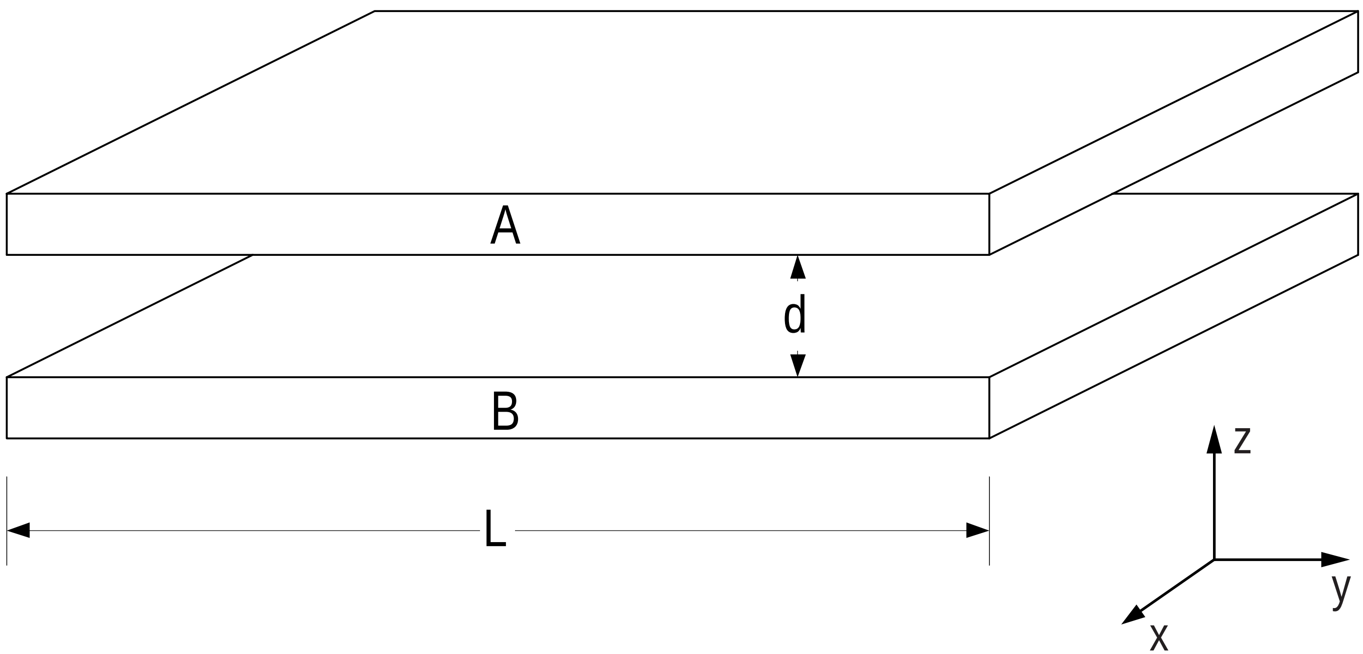


Vertically shaken layer of tiny balls (layer depth 1.2mm, ball diameter 0.15-0.18mm) [From Melo, Umbanhowar, and Swinney]


R
Driving
Perturbations grow
R_c
Perturbations decay
Equilibrium


R
Driving
Perturbations
grow
R_c
Perturbations
decay
Equilibrium

Pattern formation occurs in a spatially extended system when the growing perturbation about the spatially uniform state has spatial structure (a mode with nonzero wave vector).


A
d
B
L
z
y
x

Dynamical Equations

I shall confine my discussion to systems far from equilibrium that are macroscopic and continuous.

These are defined by dynamical equations that

- Reflect the laws of thermodynamics and the return to (local) equilibrium
- Are the familiar phenomenological equations

Leads us to the study of nonlinear, deterministic, PDEs

Equations for Convection (Boussinesq)

$$\sigma^{-1}(\partial_t + \mathbf{v}\cdot\nabla)\,\mathbf{v} = -\nabla p + R T\hat{\mathbf{z}} + \nabla^2\mathbf{v}$$

$$(\partial_t + \mathbf{v}\cdot\nabla)\,T = \nabla^2 T$$

$$\nabla\cdot\mathbf{v} = 0$$

Boundary conditions

$$\mathbf{v} = 0 \quad \text{at} \quad z = 0, 1$$

$$T = \begin{cases} 1 & \text{at} \quad z = 0 \\ 0 & \text{at} \quad z = 1 \end{cases}$$

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Conducting (no pattern) solution: $\mathbf{v} = 0$, $T = 1 - z$

A first approach to patterns: linear stability analysis

1. Find equations of motion of the physical variables $\mathbf{u}(x, y, z, t)$
2. Find the *uniform* base solution $\mathbf{u}_b(z)$ *independent* of x, y, t
3. Focus on deviation from $\mathbf{u}_b$

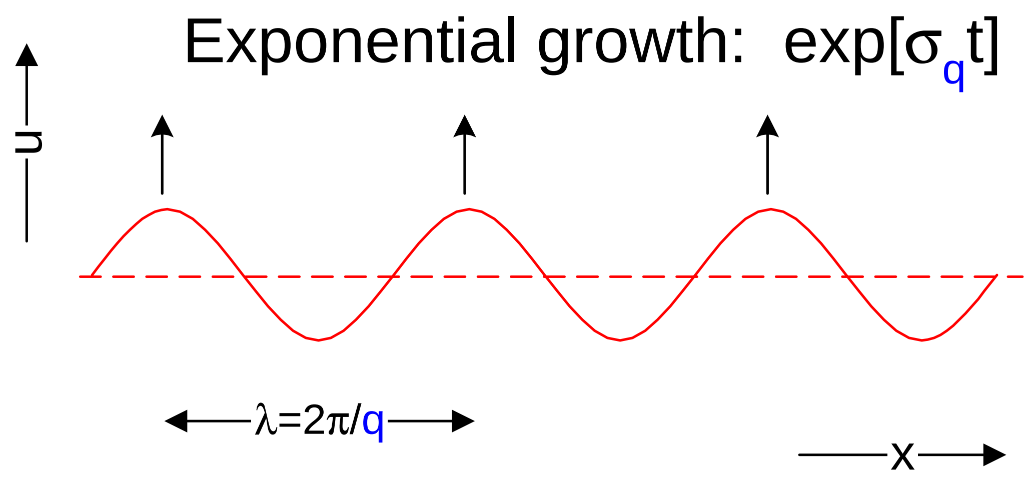
$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_b(z) + \delta\mathbf{u}(\mathbf{x}, t)$$

4. Linearize equations about $\mathbf{u}_b$, i.e. substitute into equations of part (1) and keep all terms with just one power of $\delta\mathbf{u}$. This will give an equation of the form

$$\partial_t \delta\mathbf{u} = \hat{\mathbf{L}}\, \delta\mathbf{u}$$

   where $\hat{\mathbf{L}}$ may involve $\mathbf{u}_b$ and include spatial derivatives acting on $\delta\mathbf{u}$
5. Since $\hat{\mathbf{L}}$ is independent of x, y, t we can find solutions

$$\delta\mathbf{u}_{\mathbf{q}}(\mathbf{x}_\perp, z, t) = \mathbf{u}_{\mathbf{q}}(z)\, e^{i\mathbf{q}\cdot\mathbf{x}_\perp}\, e^{\sigma_{\mathbf{q}} t}$$



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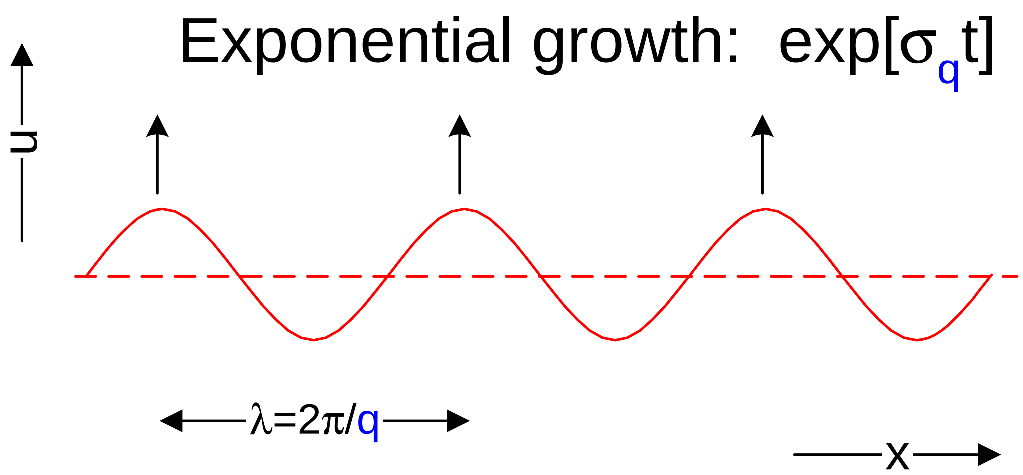

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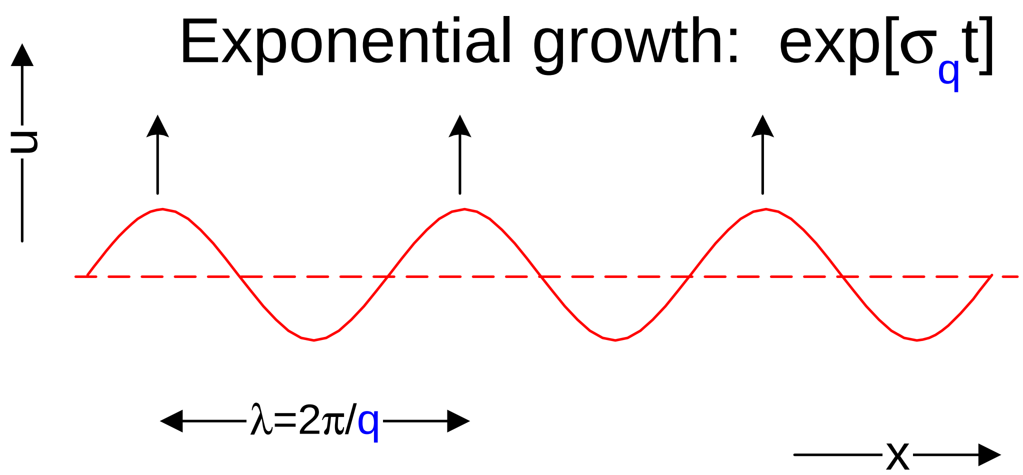
Re $\sigma_{\mathbf{q}}$ gives exponential growth or decay



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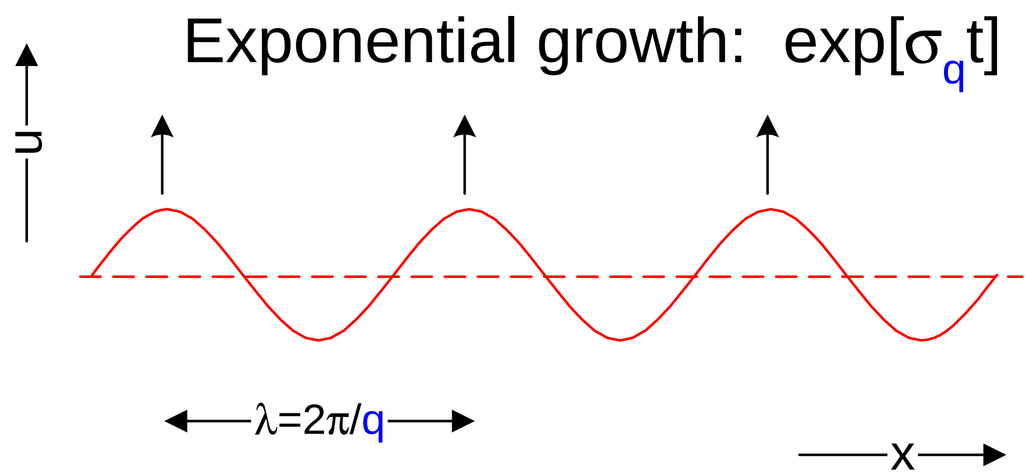
Re $\sigma_{\mathbf{q}}$ gives exponential growth or decay

Im $\sigma_{\mathbf{q}} = -\omega_{\mathbf{q}}$ gives oscillations, waves $e^{i(\mathbf{q}\cdot\mathbf{x}_\perp - \omega_{\mathbf{q}} t)}$

$$\begin{aligned} \text{Im } \sigma_{\mathbf{q}} = 0 \quad &\Longrightarrow \quad \text{Stationary instability} \\ \text{Im } \sigma_{\mathbf{q}} \neq 0 \quad &\Longrightarrow \quad \text{Oscillatory instability} \end{aligned}$$



$$\delta \mathbf{u_q}(\mathbf{x}_\perp, z, t) = \mathbf{u_q}(z)\, e^{i\,\mathbf{q}\cdot\mathbf{x}_\perp}\, e^{\sigma_\mathbf{q} t}$$

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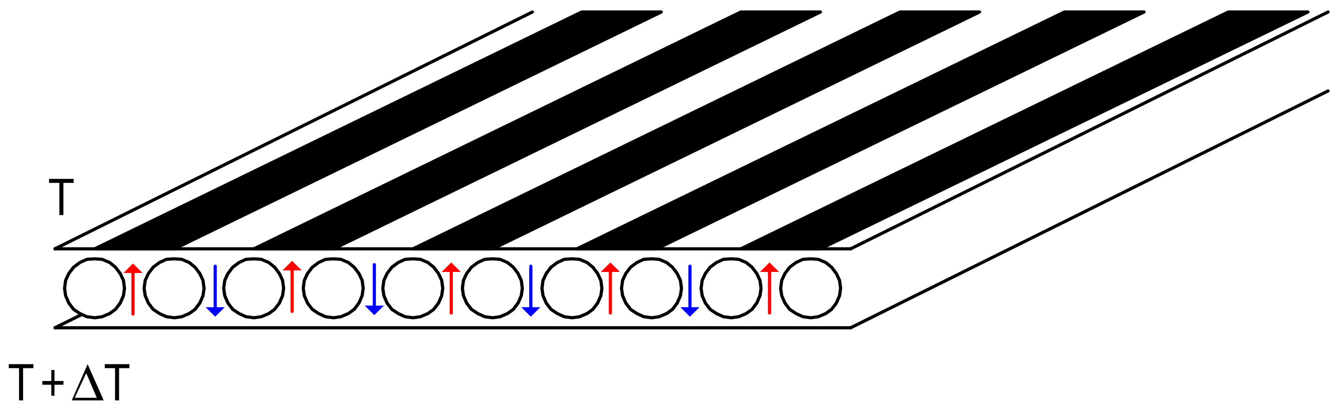
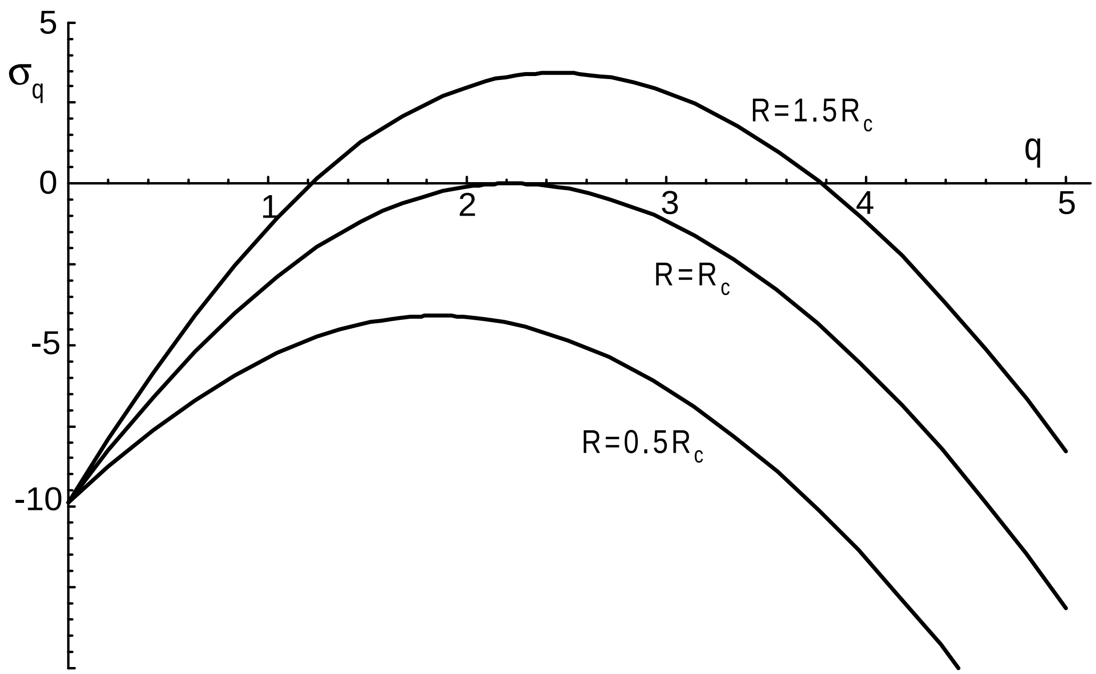
For this lecture I will look at the case of stationary instability

Rayleigh's calculation



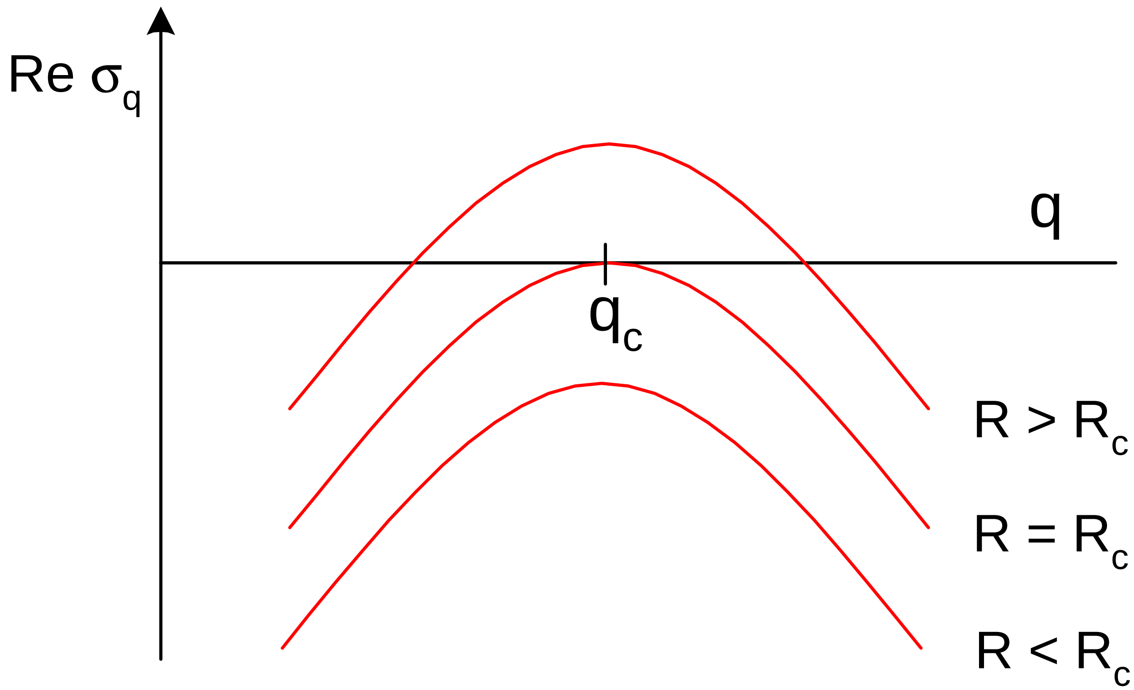
$$\delta T_q(x,z) = \left(q^2+\pi^2\right)^2 \cos(\pi z)\cos(qx),$$
$$\delta w_q(x,z) = q^2 \cos(\pi z)\cos(qx),$$
$$\delta u_q(x,z) = -i\pi q \sin(\pi z)\sin(qx).$$

Rayleigh's calculation



$$(\sigma^{-1}\sigma_q + \pi^2 + q^2)(\sigma_q + \pi^2 + q^2) - Rq^2/(\pi^2 + q^2) = 0$$

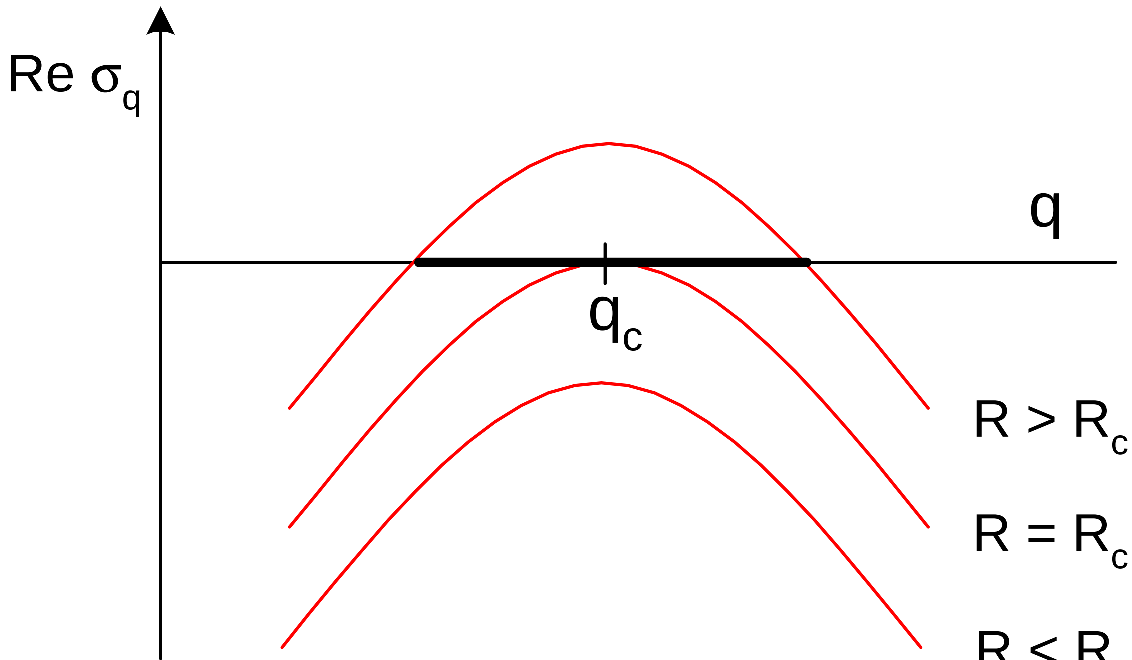
Parabolic approximation near maximum



For R near R_c and q near q_c

$$\mathrm{Re}\ \sigma_{\mathbf{q}} = \tau_0{}^{-1}[\varepsilon - \xi_0{}^2\,(q - q_c)^2] \qquad \text{with} \qquad \varepsilon = \frac{R - R_c}{R_c}$$

Parabolic approximation near maximum



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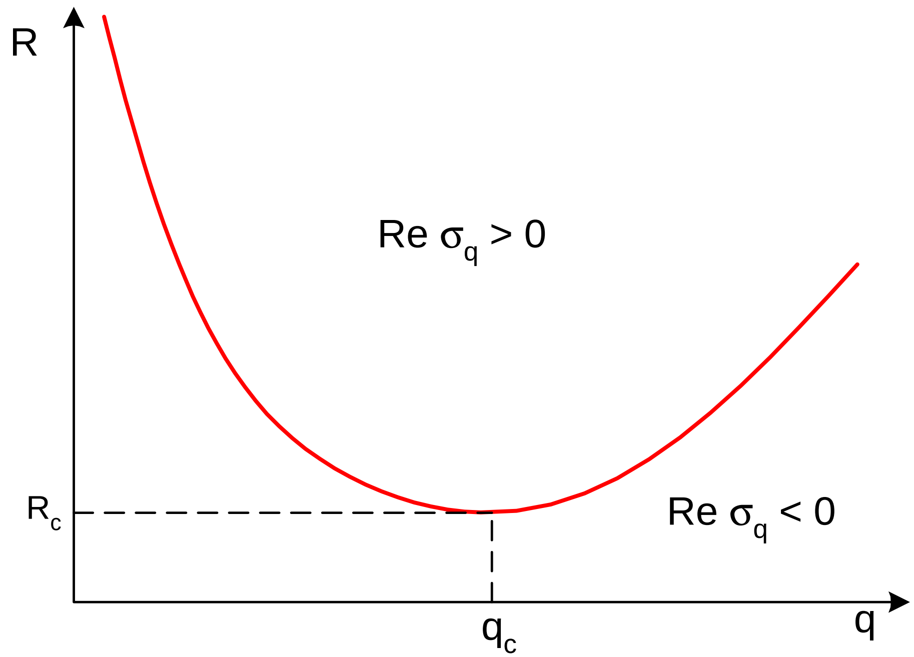
Neutral stability curve



Setting Re $\sigma_{\mathbf{q}} = 0$ defines the neutral stability curve $R = R_c(q)$

$$\text{Rayleigh :} \qquad R_c(q) = \frac{(q^2+\pi^2)^3}{q^2} \Rightarrow \qquad R_c = \frac{27\pi^4}{4},\ q_c = \frac{\pi}{\sqrt{2}}$$

Linear stability theory is often a useful first step in understanding pattern formation:

- Often is quite easy to do either analytically or numerically
- Displays the important physical processes
- Gives the length scale of the pattern formation $1/q_c$

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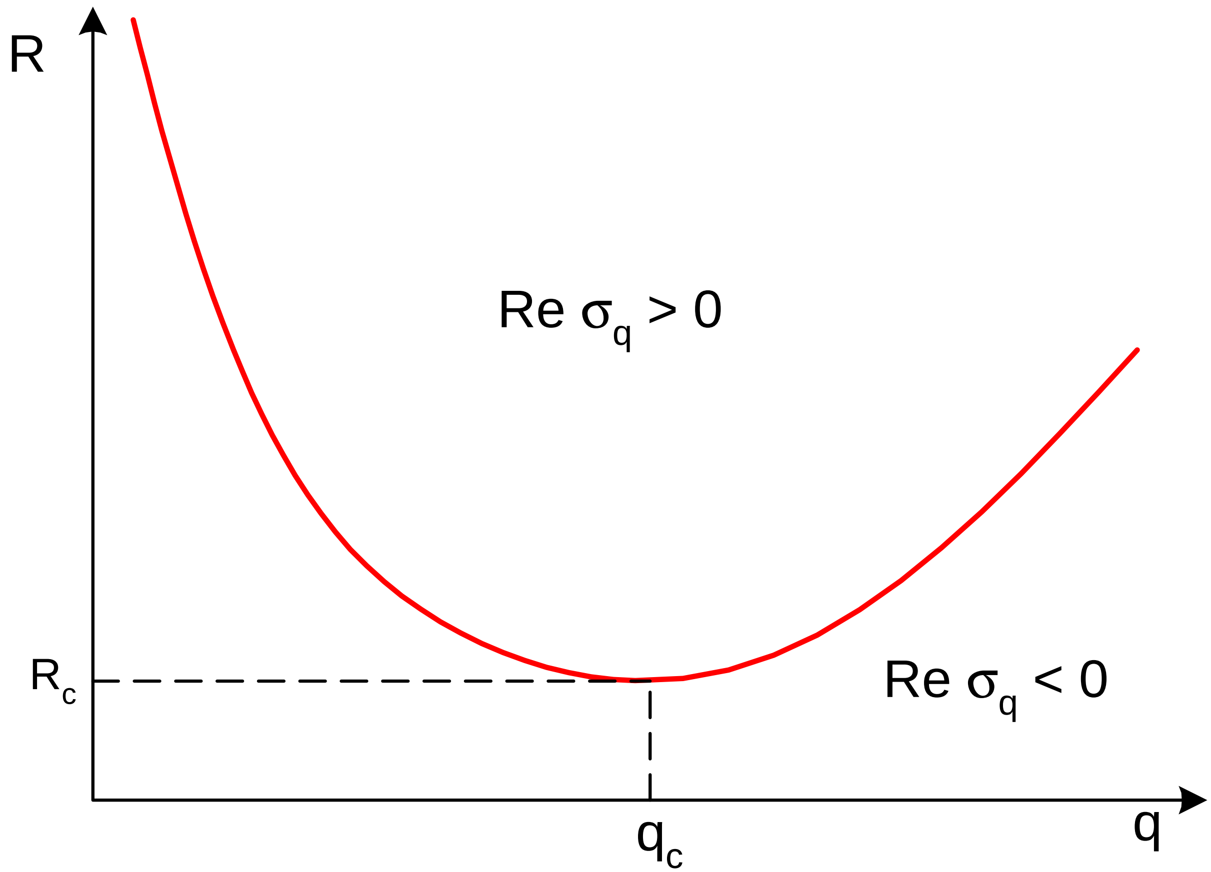
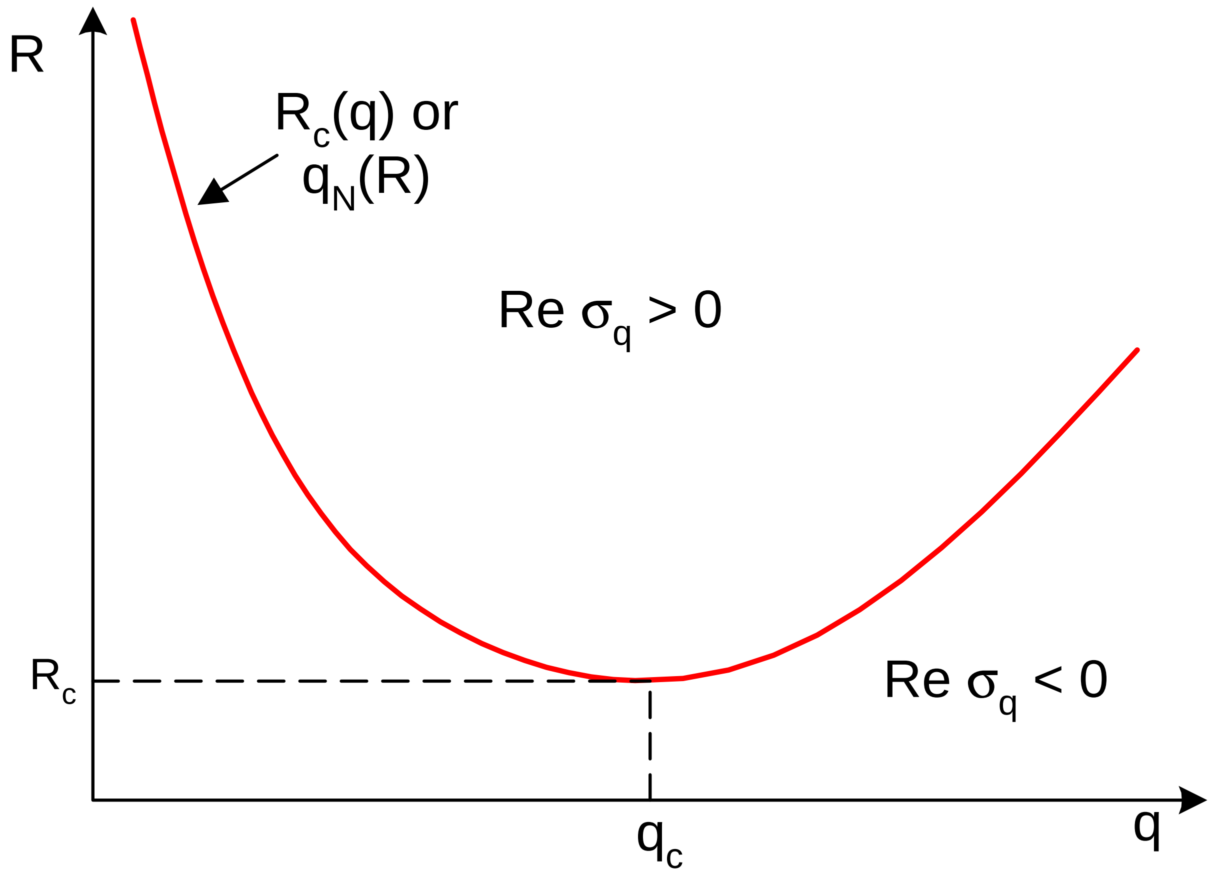
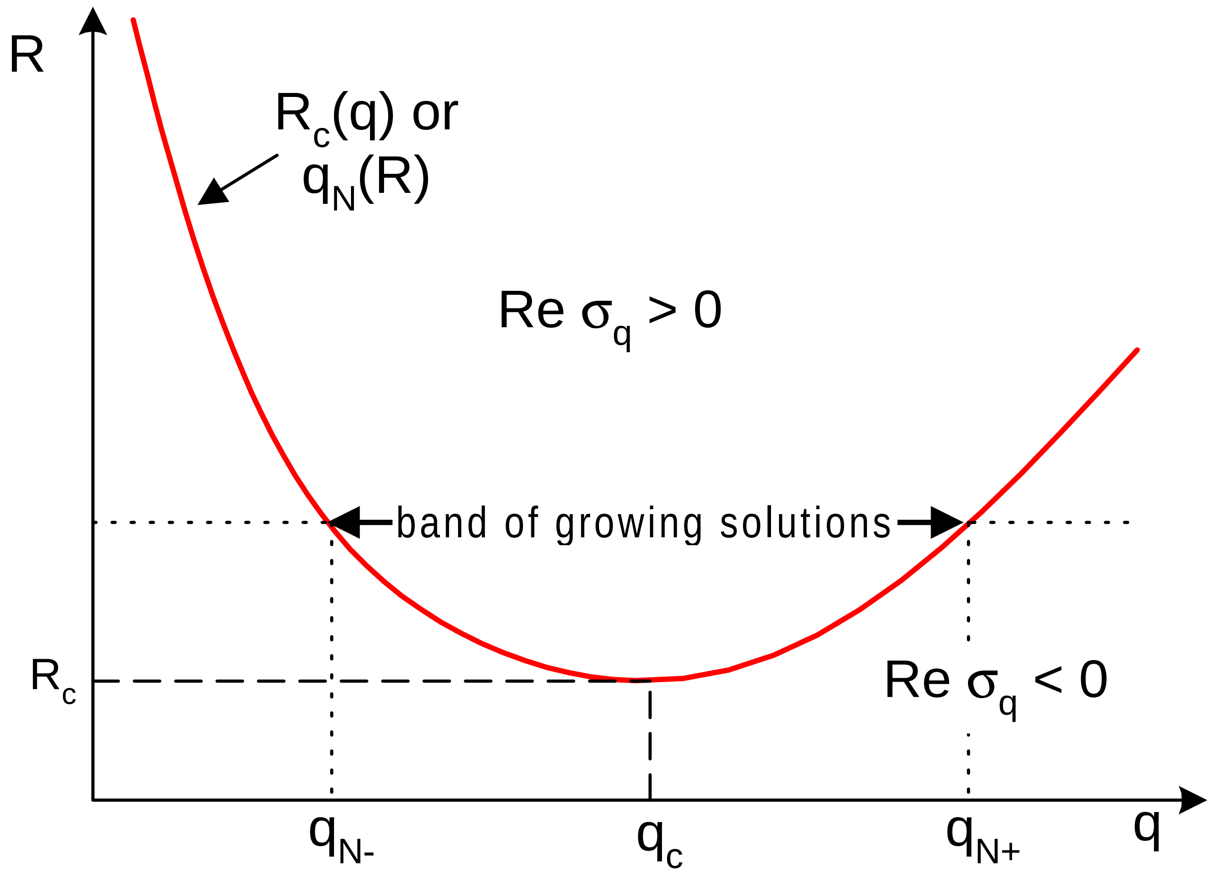
But:

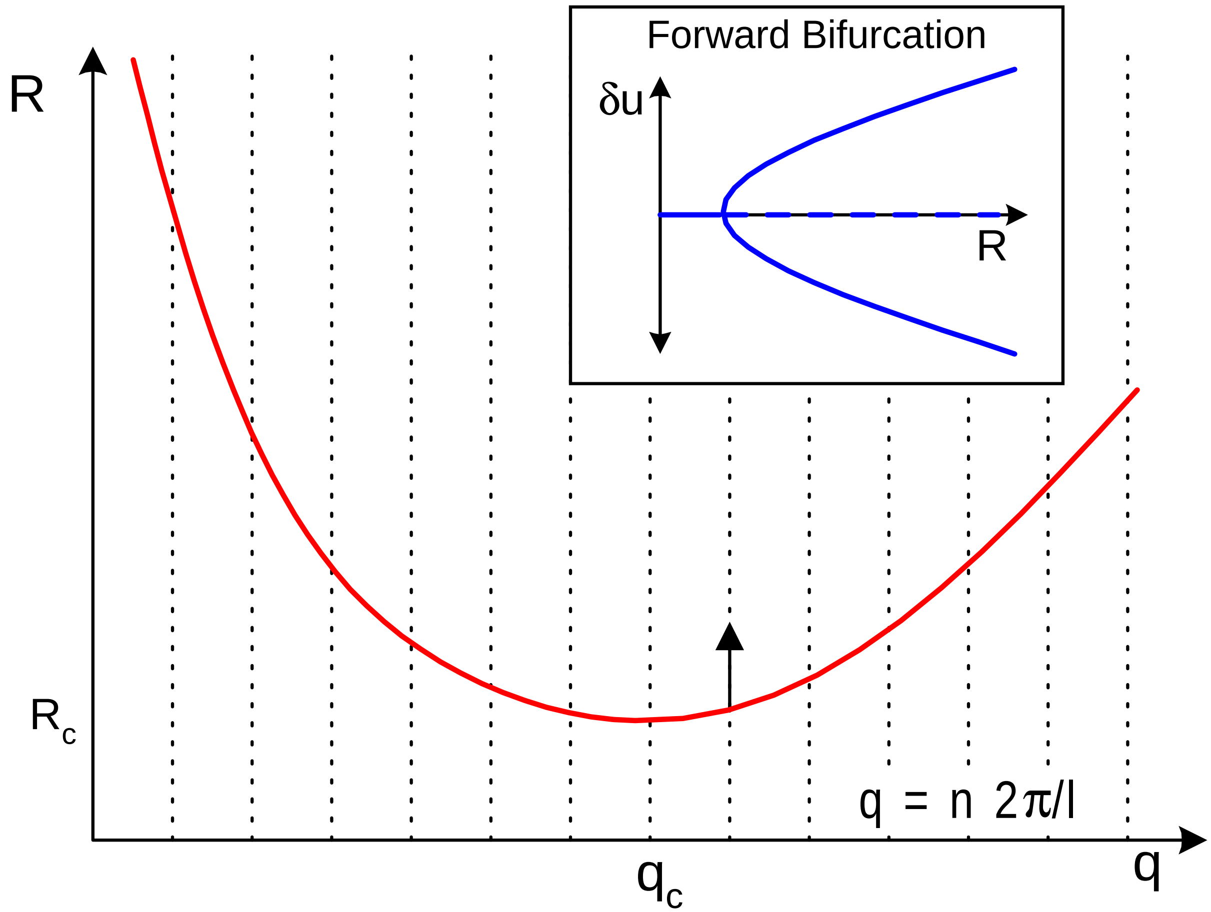
- Leaves us with unphysical exponentially growing solutions

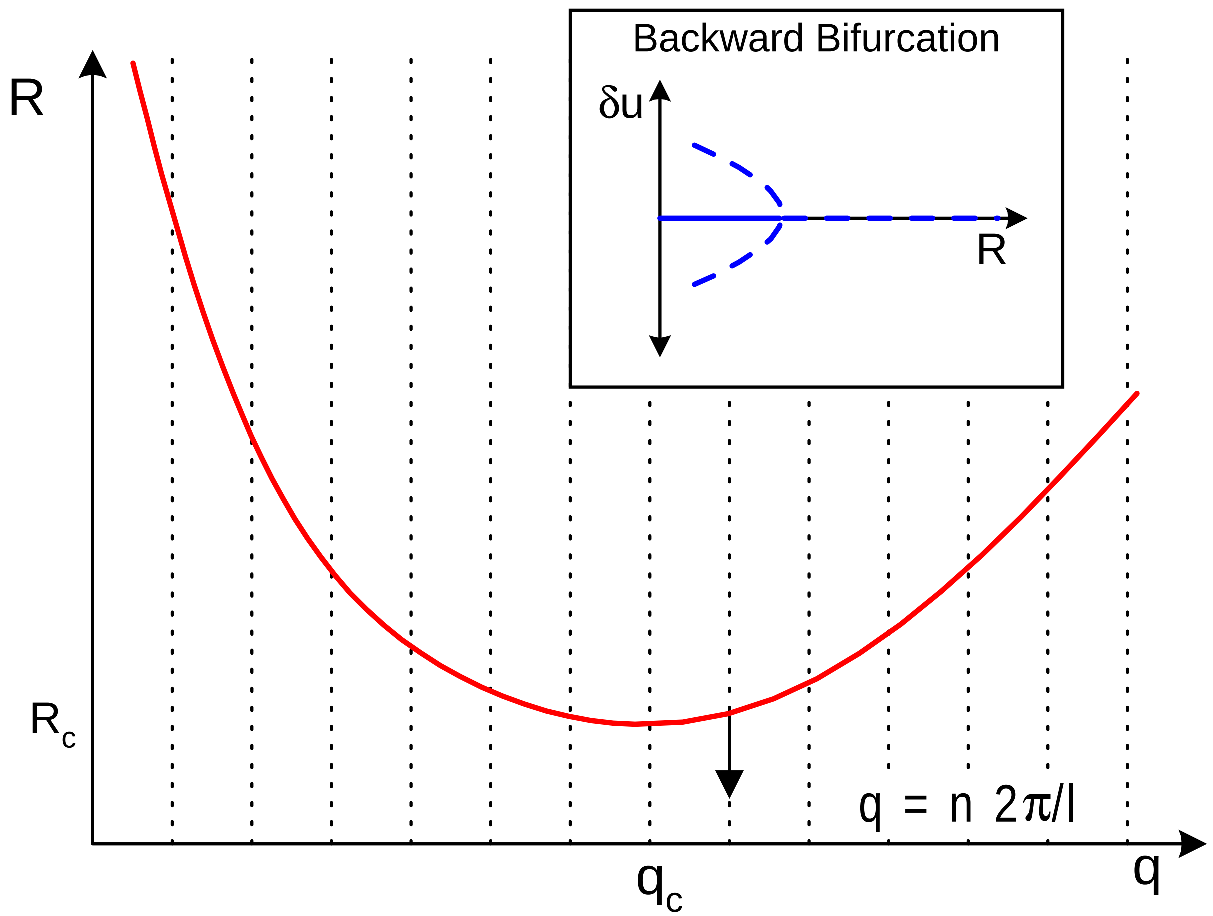
Nonlinearity

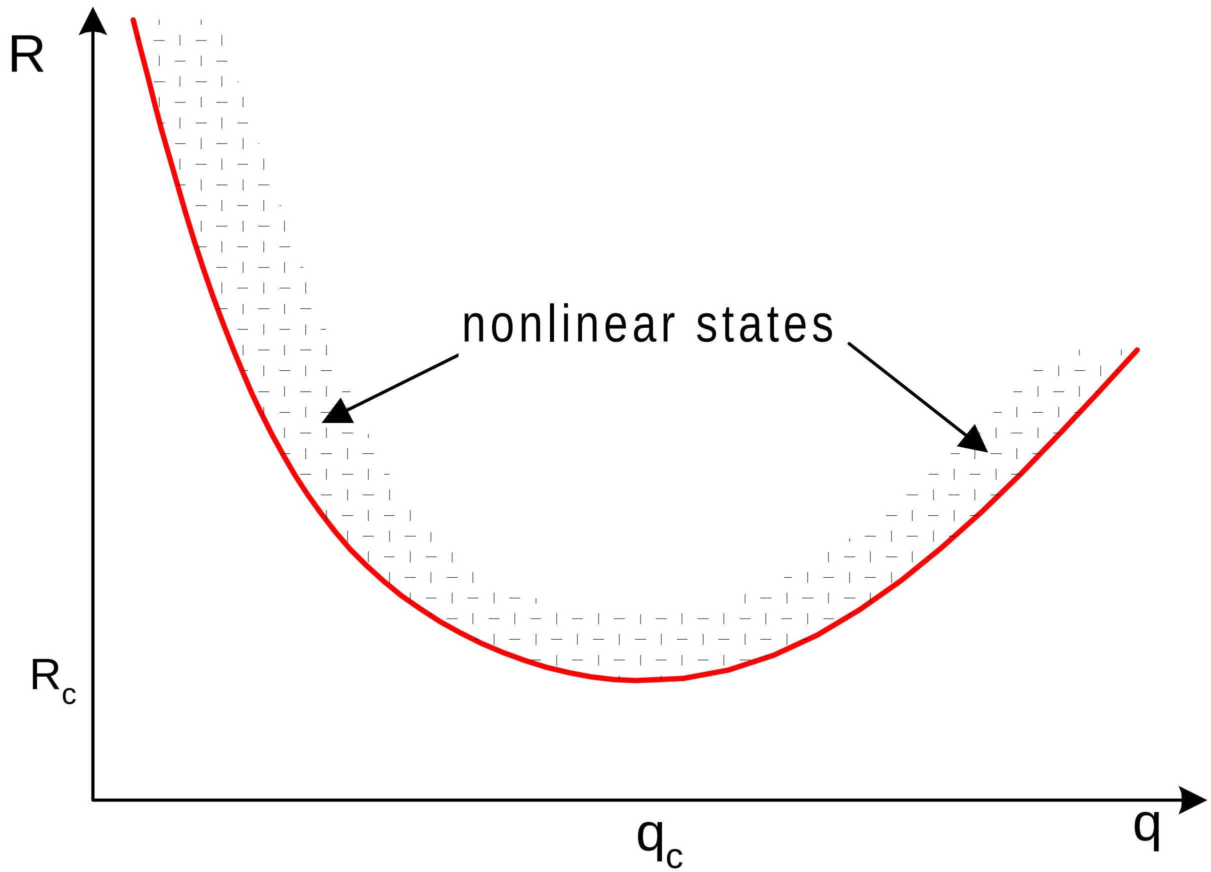

R
Re σq > 0
Rc
Re σq < 0
qc
q

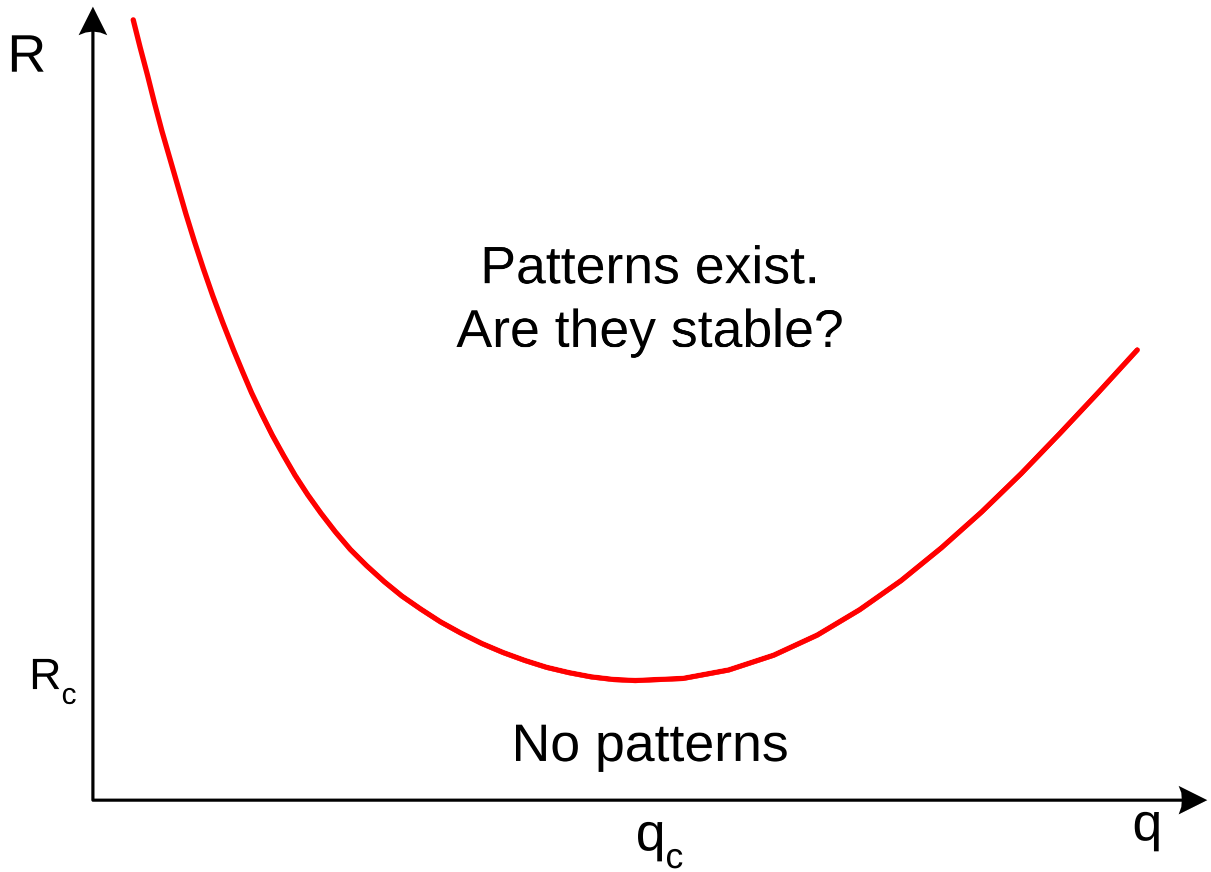

R
Rc(q) or
qN(R)
Re σq > 0
Rc
Re σq < 0
qc
q

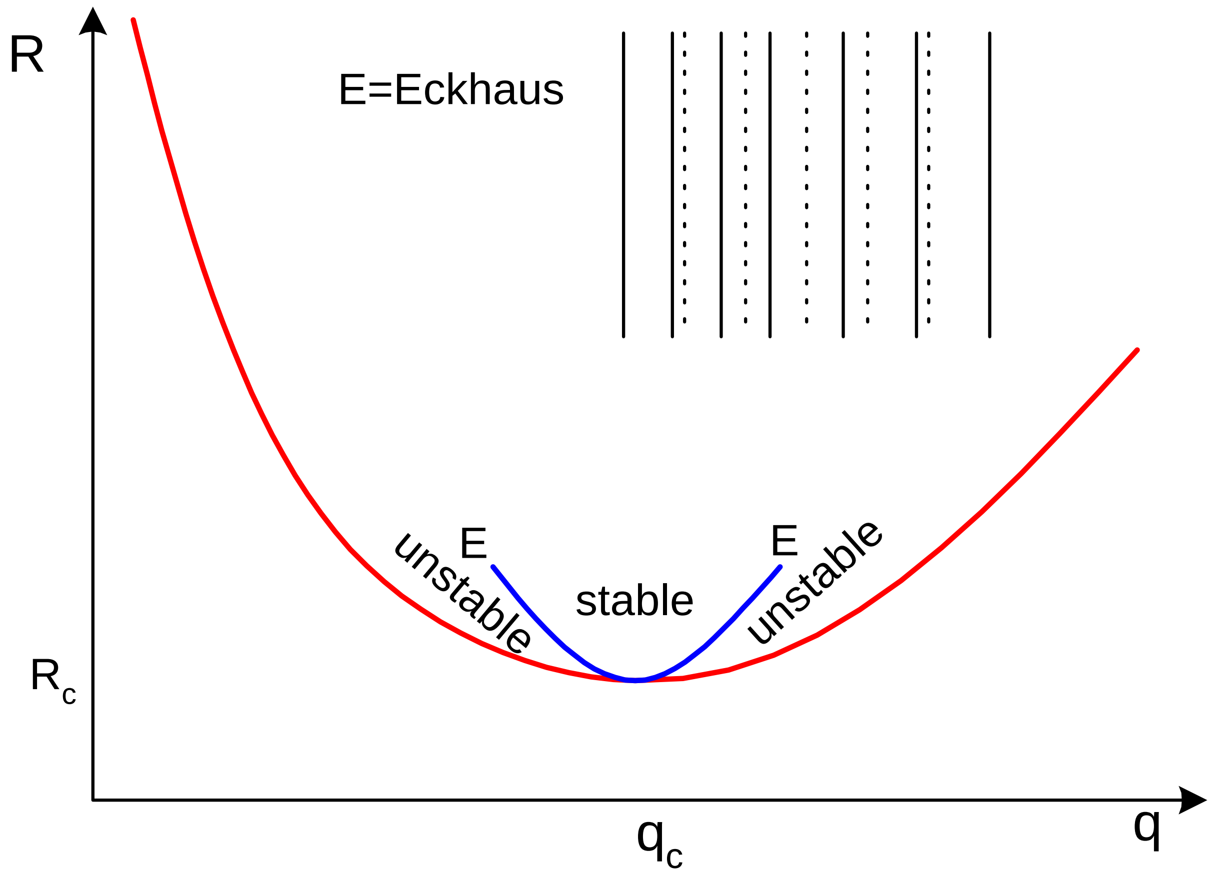

R
Rc(q) or
qN(R)
Re σq > 0
band of growing solutions
Rc
Re σq < 0
qN-
qc
qN+
q

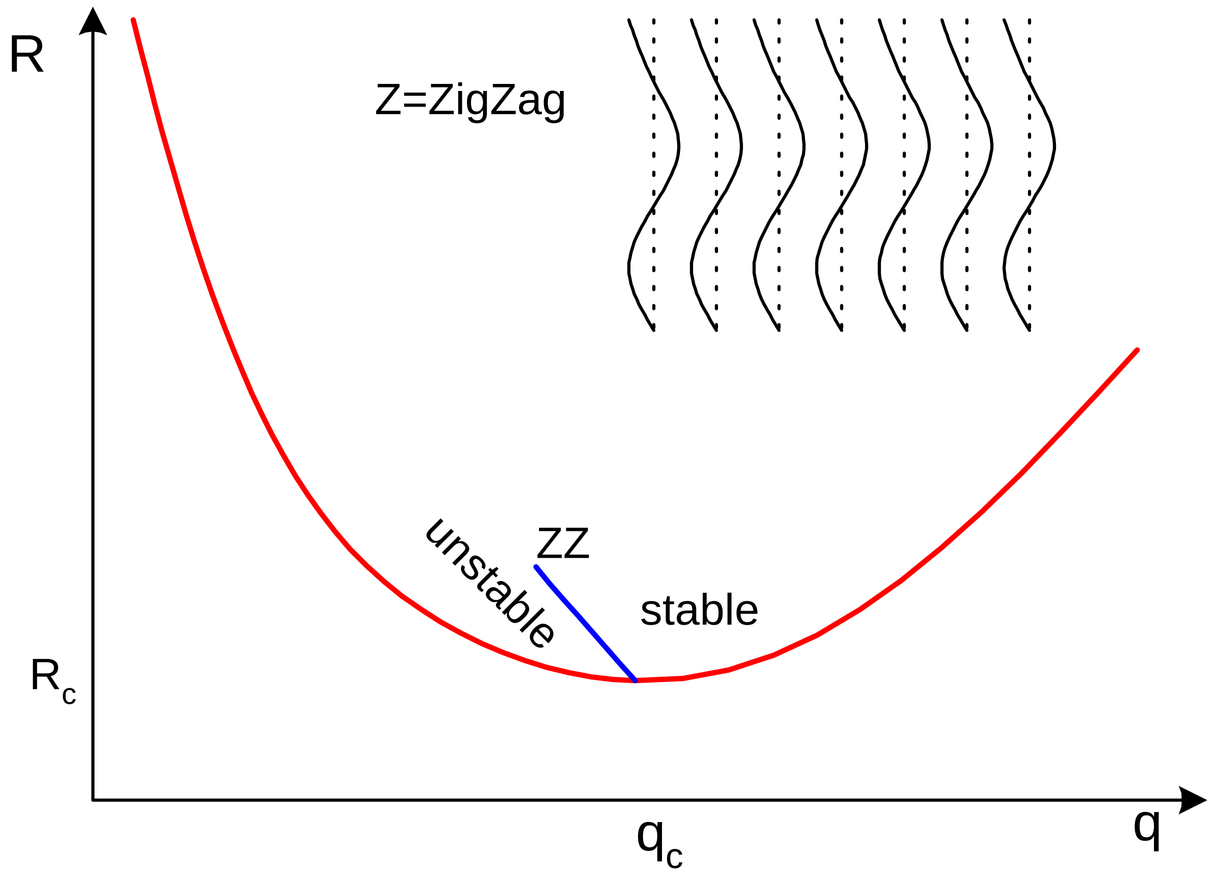

Forward Bifurcation
δu
R
R
R_c
q = n 2π/l
q_c
q

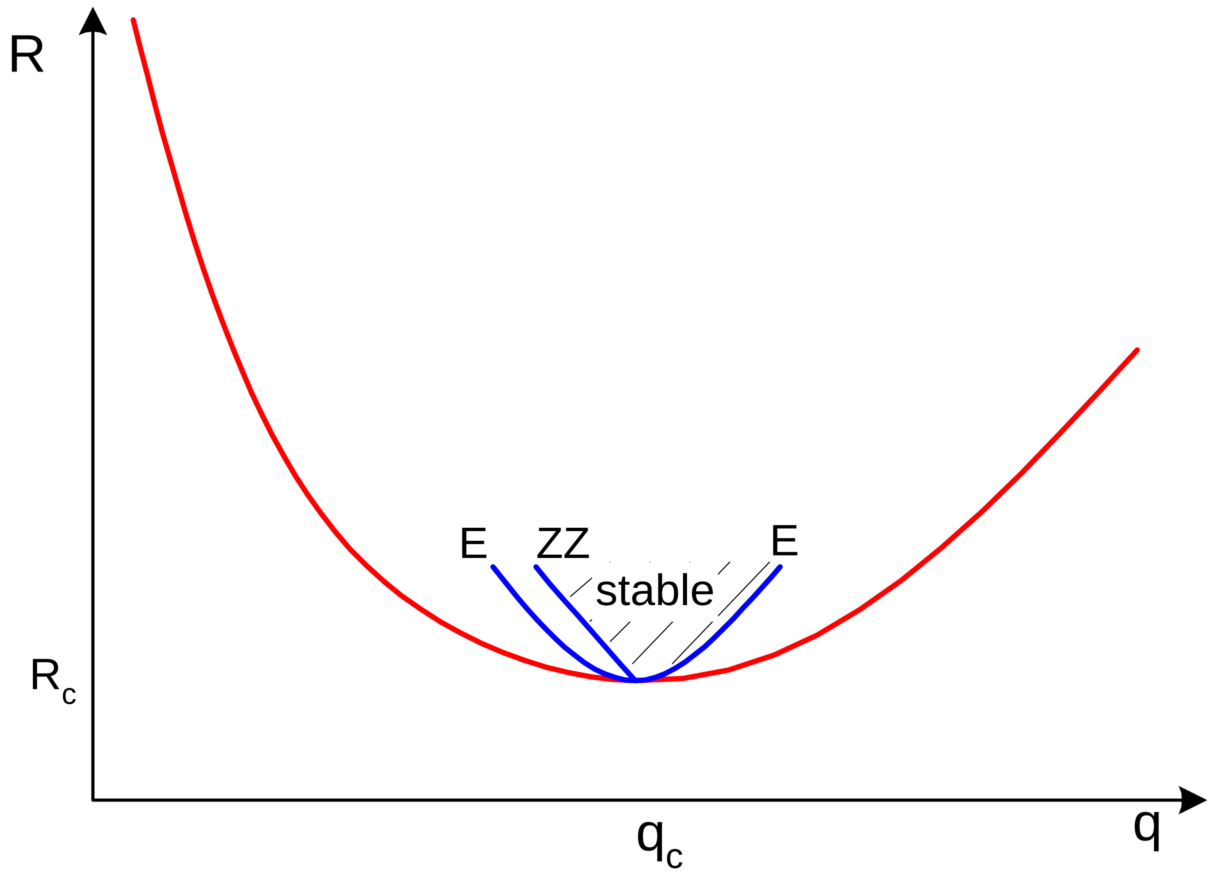

Backward Bifurcation
δu
R
R
R_c
q = n 2π/l
q_c
q

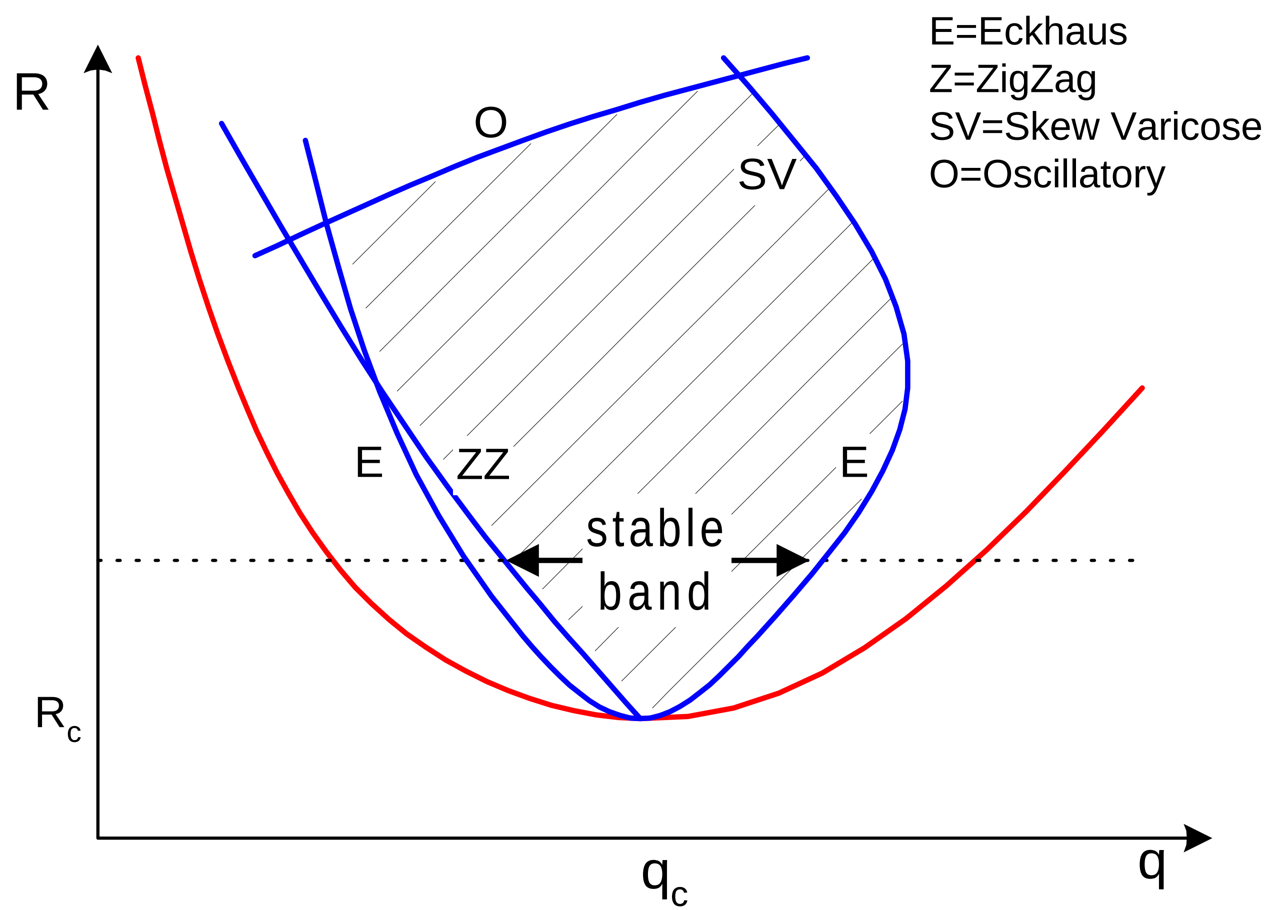

R
nonlinear states
Rc
qc
q

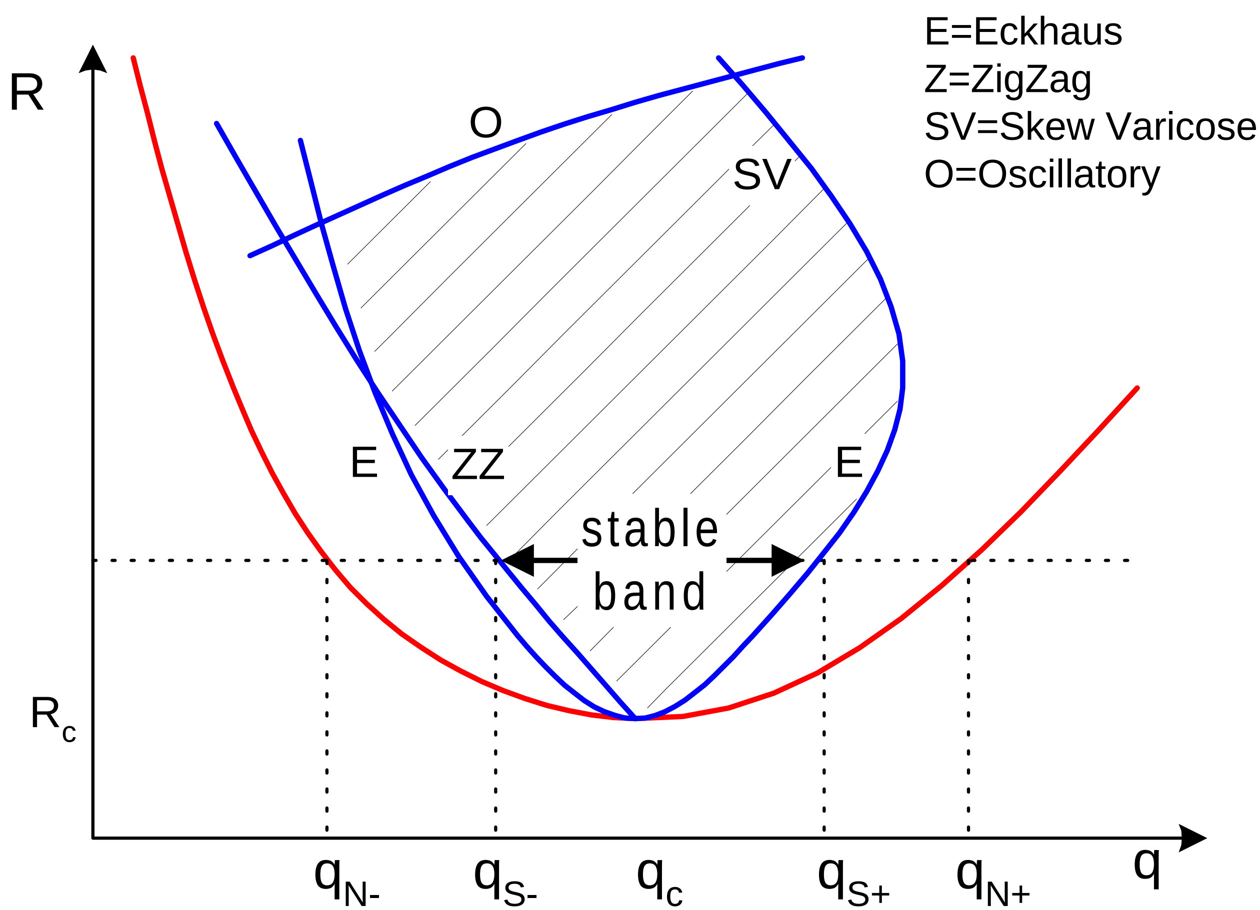

R
Patterns exist.
Are they stable?
Rc
No patterns
qc
q


R
E=Eckhaus
E
E
unstable
stable
unstable
R_c
q_c
q


R
Z=ZigZag
unstable
ZZ
stable
R_c
q_c
q


R
E
ZZ
E
stable
R_c
q_c
q


E=Eckhaus
Z=ZigZag
SV=Skew Varicose
O=Oscillatory
R
O
SV
E
ZZ
E
stable
band
R_c
q_c
q


R
O
SV
E
ZZ
E
stable
band
R_c
q_{N-}
q_{S-}
q_c
q_{S+}
q_{N+}
q
E=Eckhaus
Z=ZigZag
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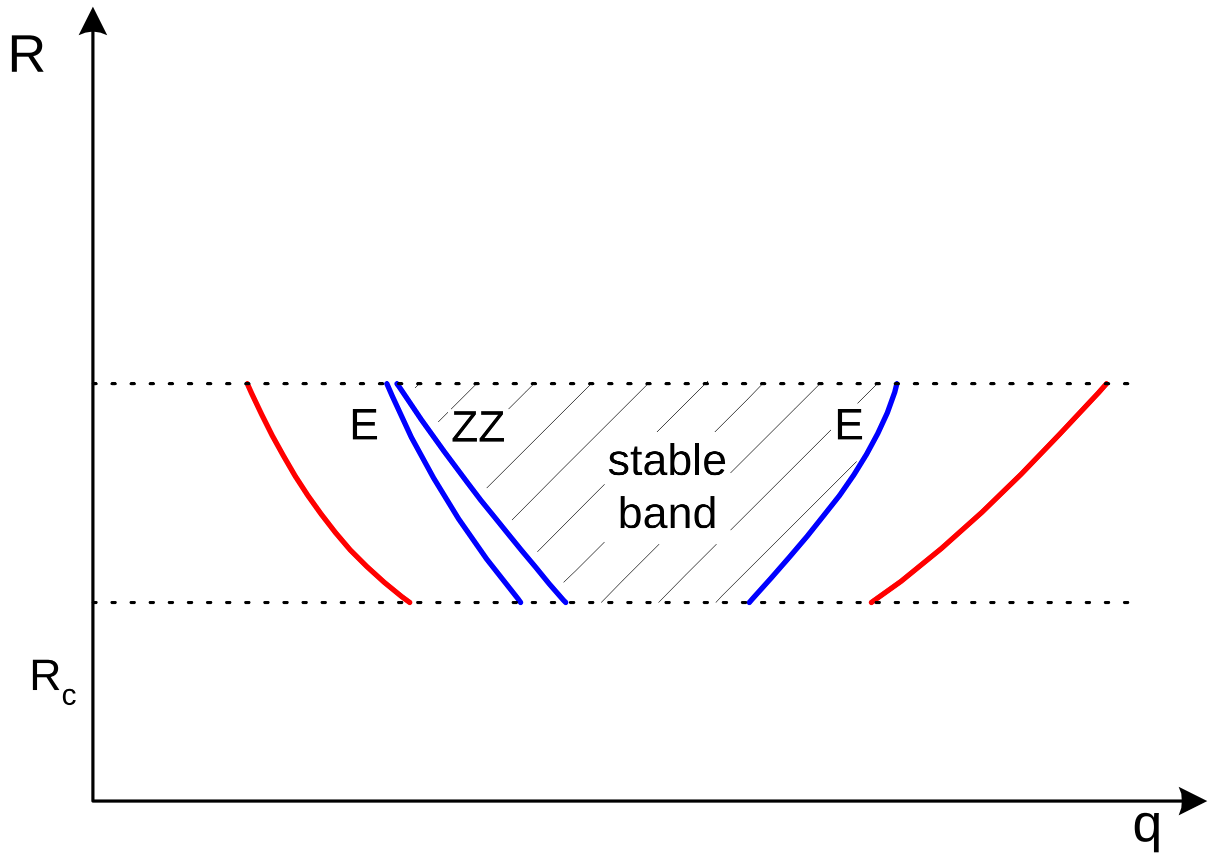
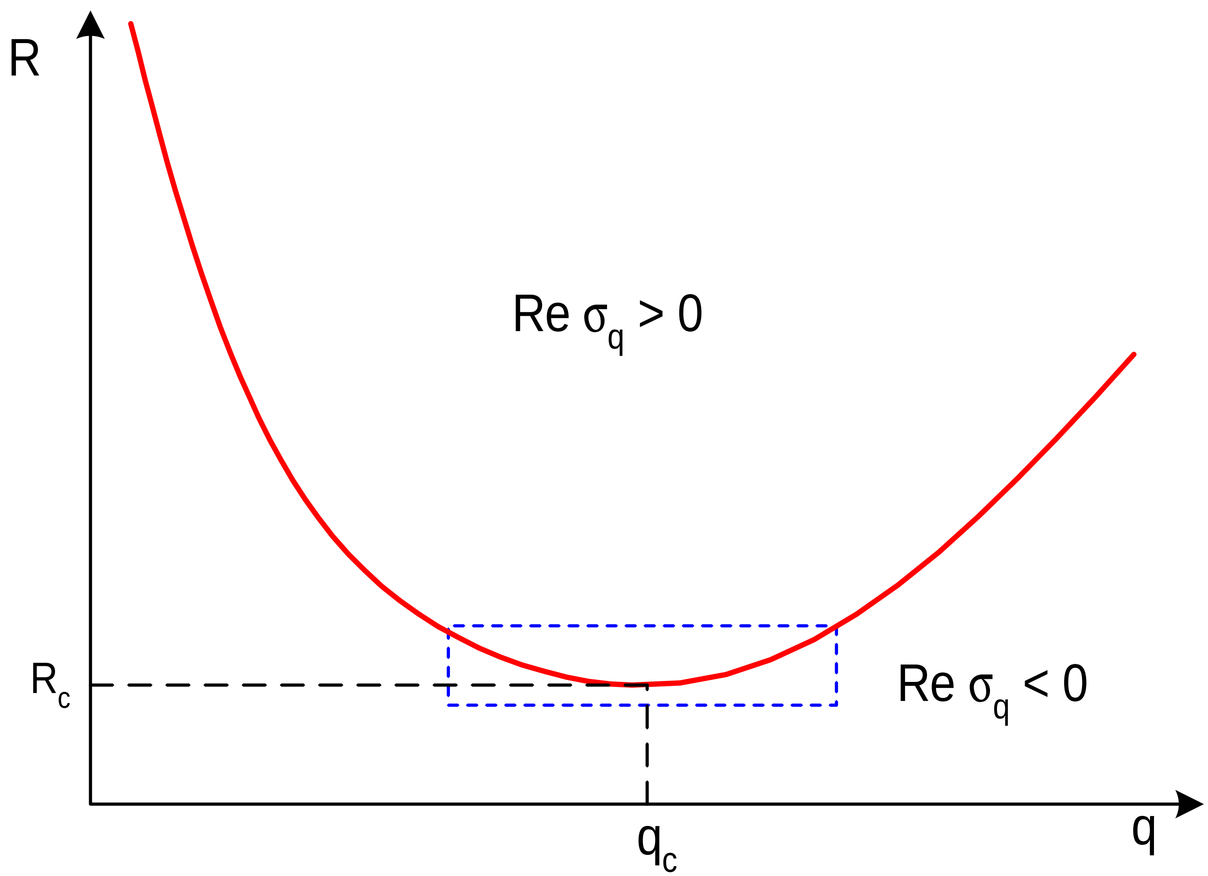

R
E
ZZ
stable
band
E
Rc
q

Tools for the Nonlinear Problem

Amplitude Equations

Systematic approach for describing weakly nonlinear solutions near onset



Linear onset solution

$$\delta \mathbf{u}_{\mathbf{q}}(\mathbf{x}_\perp, z, t) \quad = \quad \left[a_0 e^{i(\mathbf{q}-\mathbf{q}_c)\cdot\mathbf{x}_\perp}\, e^{\mathrm{Re}\,\sigma_{\mathbf{q}} t}\right] \quad \times \quad \left[\mathbf{u}_{\mathbf{q}}(z)\, e^{i\mathbf{q}_c\cdot\mathbf{x}_\perp}\right] \quad + \quad c.c.$$

Small terms near onset — Onset solution

Linear onset solution

$$\delta \mathbf{u}_{\mathbf{q}}(\mathbf{x}_\perp, z, t) \quad = \quad \underbrace{\left[a_0 e^{i(\mathbf{q}-\mathbf{q}_c)\cdot\mathbf{x}_\perp} e^{\mathrm{Re}\,\sigma_{\mathbf{q}} t}\right]}_{\text{Small terms near onset}} \quad \times \quad \underbrace{\left[\mathbf{u}_{\mathbf{q}}(z)\, e^{i\mathbf{q}_c\cdot\mathbf{x}_\perp}\right]}_{\text{Onset solution}} \quad + \quad c.c.$$

Weakly nonlinear, slowly modulated, solution

$$\delta \mathbf{u}(\mathbf{x}_\perp, z, t) \quad \approx \quad \underbrace{A(\mathbf{x}_\perp, t)}_{\text{Complex amplitude}} \quad \times \quad \underbrace{\left[\mathbf{u}_{\mathbf{q}_c}(z)\, e^{i\mathbf{q}_c\cdot\mathbf{x}_\perp}\right]}_{\text{Onset solution}} \quad + \quad c.c.$$

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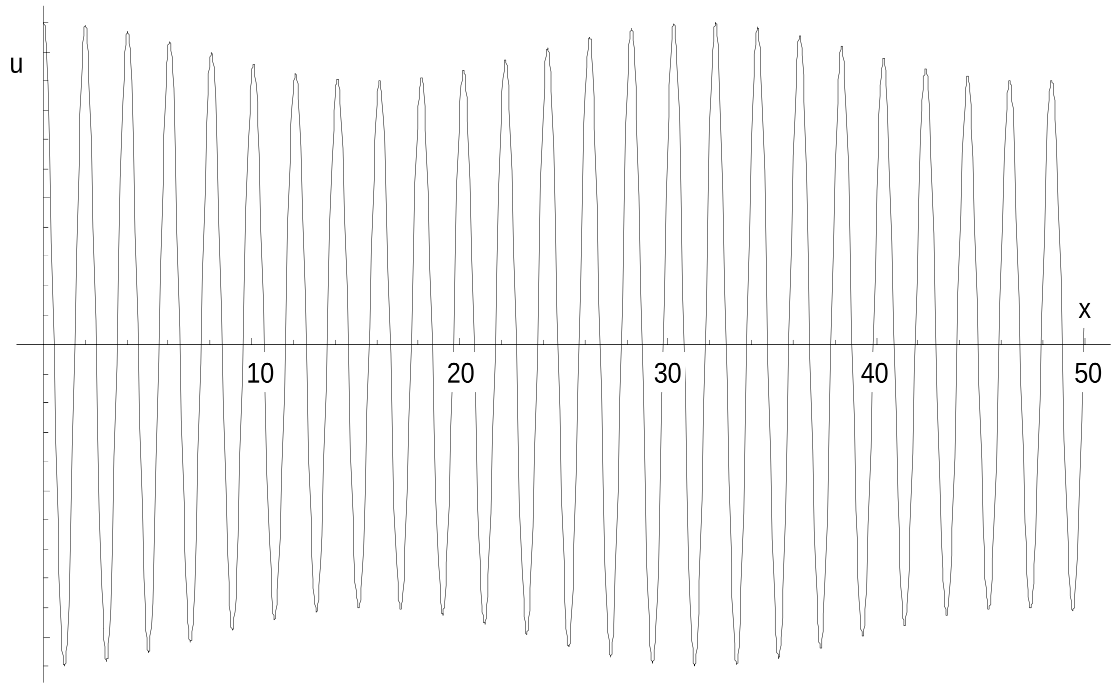
Substituting into the dynamical equations gives the amplitude equation, which in 1d [$\mathbf{q}_c = q_c\hat{\mathbf{x}}$, $A = A(x,t)$] is

$$\tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \qquad \varepsilon = \frac{R - R_c}{R_c}$$

Pictorially

A convection pattern that varies gradually in space



$$u \propto \mathrm{Re}[A(x)e^{iq_c x}]$$

$$q_c = 3.117; \qquad A(x) = 1 + 0.1\cos(0.2x)$$

Complex Amplitude

Magnitude and phase of A play very different roles

$$A(x, y, t) = a(x, y, t)e^{i\theta(x,y,t)}$$

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$$\delta\mathbf{u}(\mathbf{x}_\perp, z, t) = \, ae^{i\theta} \, \times \, e^{iq_c x}\mathbf{u}_{\mathbf{q}_c}(z) \, + \, c.c.$$

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- magnitude $a = |A|$ gives strength of disturbance

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- phase $\delta\theta$ gives shift of pattern (by $\delta x = \delta\theta/q_c$)

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- magnitude $a = |A|$ gives strength of disturbance
- phase $\delta\theta$ gives shift of pattern (by $\delta x = \delta\theta/q_c$)— symmetry!
- x-gradient $\partial_x\theta$ gives change of wave number $q = q_c + \partial_x\theta$
  $A = ae^{ikx}$ corresponds to $q = q_c + k$

Complex Amplitude

Magnitude and phase of A play very different roles

$$A(x, y, t) = a(x, y, t)e^{i\theta(x,y,t)}$$

$$\delta\mathbf{u}(\mathbf{x}_\perp, z, t) = ae^{i\theta} \times e^{iq_c x}\mathbf{u}_{\mathbf{q}_c}(z) + c.c.$$

- magnitude $a = |A|$ gives strength of disturbance
- phase $\delta\theta$ gives shift of pattern (by $\delta x = \delta\theta/q_c$)— symmetry!
- x-gradient $\partial_x\theta$ gives change of wave number $q = q_c + \partial_x\theta$
  $A = ae^{ikx}$ corresponds to $q = q_c + k$
- y-gradient $\partial_y\theta$ gives rotation of wave vector through angle $\partial_y\theta/q_c$ (plus $O[(\partial_y\theta)^2]$ change in wave number)

The amplitude equation describes

$$\tau_0 \partial_t A = \underbrace{\varepsilon A}_{\text{growth}} + \underbrace{\xi_0^2 \partial_x^2 A}_{\text{dispersion/diffusion}} - \underbrace{g_0 |A|^2 A}_{\text{saturation}}$$

Parameters

$$\tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A,$$

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    $A = a\, e^{i\mathbf{k}\cdot\mathbf{x}_\perp} e^{\sigma_{\mathbf{q}} t}$ gives pattern at $\mathbf{q} = \mathbf{q}_c \hat{x} + \mathbf{k}$)

$$\sigma_{\mathbf{q}} = \tau_0{}^{-1}[\varepsilon - \xi_0{}^2 (q - q_c)^2]$$

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  - ◇ g_0 by calculating nonlinear state at small ε and $q = q_c$.

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$$\tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \qquad \varepsilon = \frac{R - R_c}{R_c}$$

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Introduce scaled variables

$$x = \varepsilon^{-1/2} \xi_0 \, X$$

$$t = \varepsilon^{-1} \tau_0 \, T$$

$$A = (\varepsilon / g_0)^{1/2} \, \bar{A}$$

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Since solutions to this equation will develop on scales $X, Y, T, \bar{A} = O(1)$ this gives us scaling results for the physical length scales.

Derivation

$$\tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A, \qquad \varepsilon = \frac{R - R_c}{R_c}$$

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- Symmetry arguments: equation invariant under:
  - ◇ $A(\mathbf{x}_\perp) \to A(\mathbf{x}_\perp)e^{i\Delta}$ with Δ a constant, corresponding to a physical translation;
  - ◇ $A(\mathbf{x}_\perp) \to A^*(-\mathbf{x}_\perp)$, corresponding to inversion of the horizontal coordinates (parity symmetry);

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- Multiple scales perturbation theory (Newell and Whitehead, Segel 1969)
- Mode projection (MCC 1980)

Amplitude Equation = Ginzburg Landau equation

$$\tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A,$$

Familiar from other branches of physics:

- Good: take intuition from there
- Bad: no *really* new effects

e.g. equation is relaxational (potential, Lyapunov)

$$\tau_0 \partial_t A = -\frac{\delta V}{\delta A^*}, \qquad V = \int dx \left[-\varepsilon |A|^2 + \tfrac{1}{2} g_0 |A|^4 + \xi_0^2 |\partial_x A|^2 \right]$$

This leads to

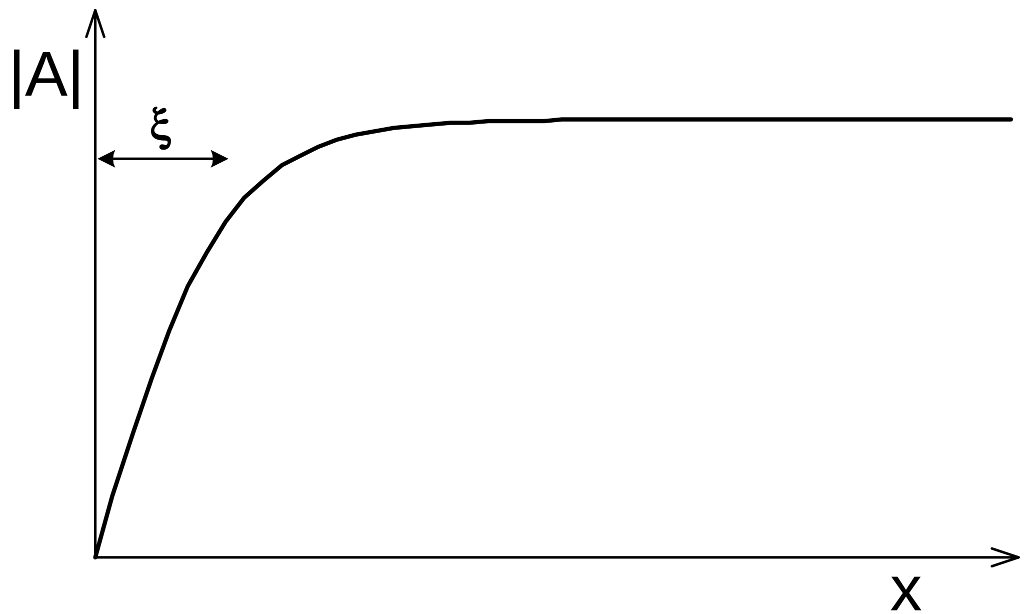
$$\frac{dV}{dt} = -\tau_0^{-1} \int dx \, |\partial_t A|^2 \leq 0$$

and dynamics runs “down hill” to a minimum of V— no chaos!

Example: one dimensional geometry with boundaries that suppress the pattern (e.g. rigid walls in a convection system)

First consider a single wall

$$\partial_T \bar{A} = \bar{A} + \partial_X^2 \bar{A} - \left|\bar{A}\right|^2 \bar{A} \qquad \bar{A}(0) = 0$$



$$\bar{A} = e^{i\theta} \tanh(X/\sqrt{2})$$

$$A = e^{i\theta}(\varepsilon/g_0)^{1/2} \tanh(x/\xi) \qquad \text{with} \qquad \xi = \sqrt{2}\varepsilon^{-1/2}\xi_0$$

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- arbitrary position of rolls
- asymptotic wave number is $k = 0$, giving $q = q_c$: no band of existence

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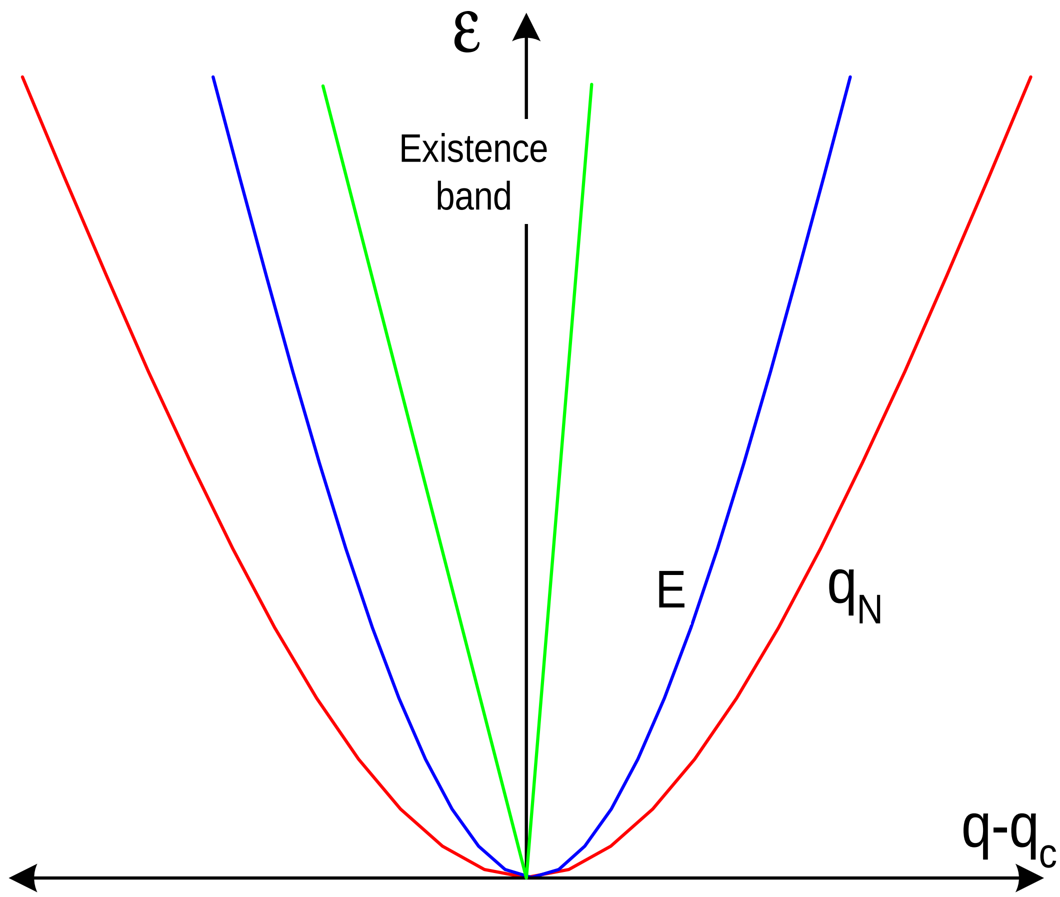
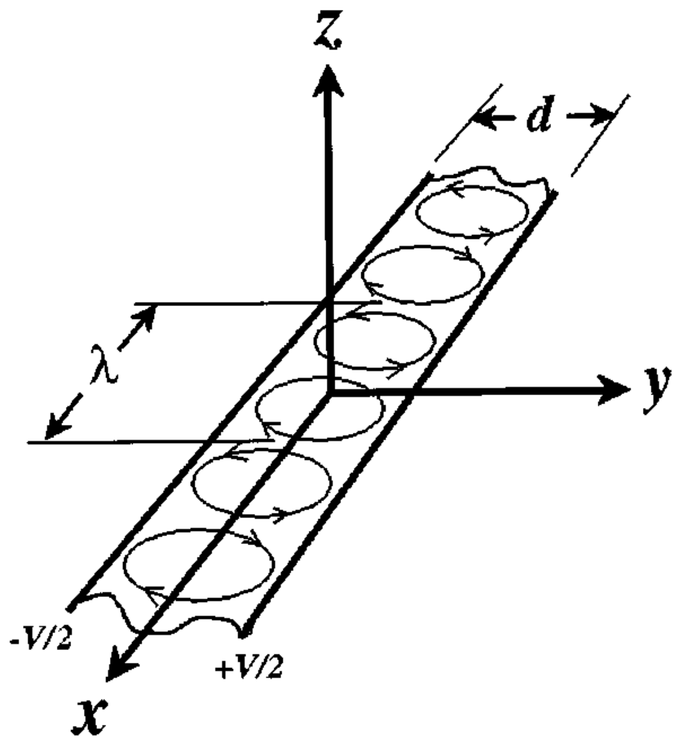
Extended amplitude equation to next order in ε (MCC, Daniels, Hohenberg, and Siggia 1980) shows

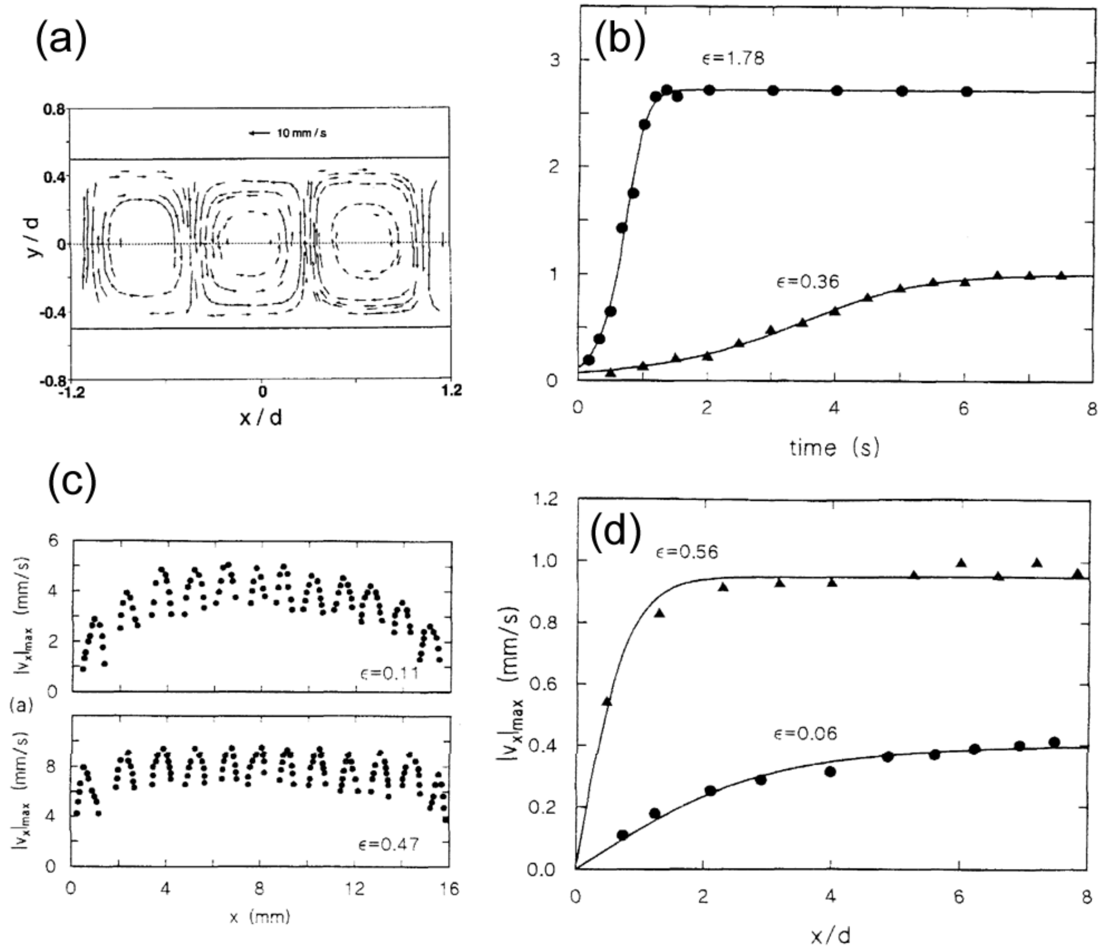
- discrete set of roll positions
- solutions restricted to a narrow $O(\varepsilon^1)$ wave number band with wave number far from the wall

$$\alpha_-\varepsilon < q - q_c < \alpha_+\varepsilon$$

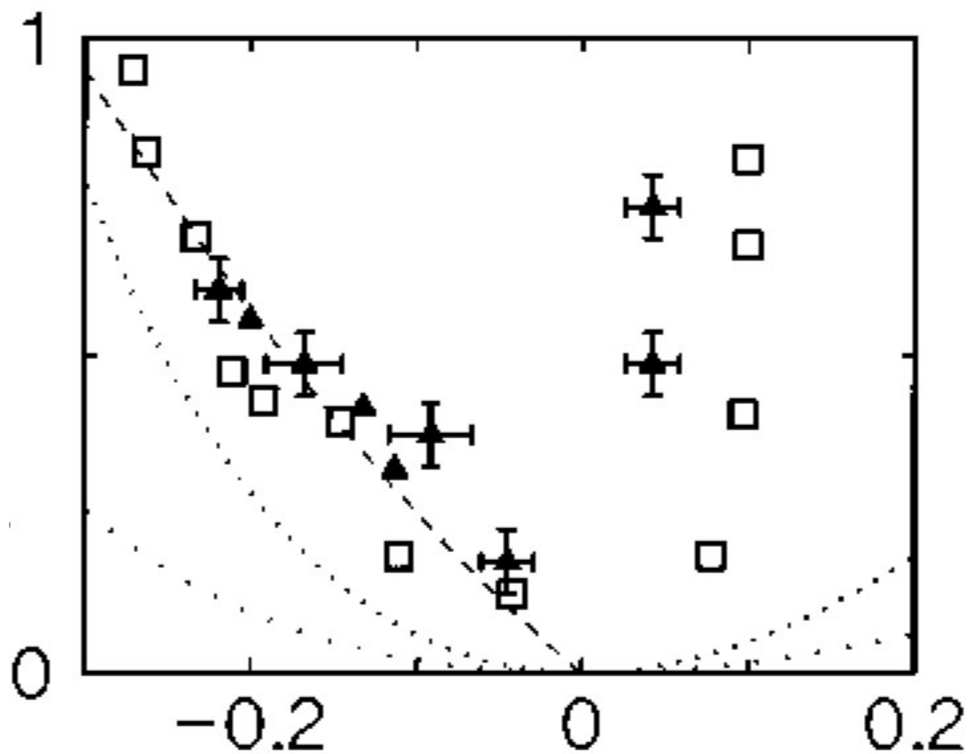

ε
Existence
band
E
q_N
$q-q_c$



V. B. Deyirmenjian, Z. A. Daya, and S. W. Morris (1997)

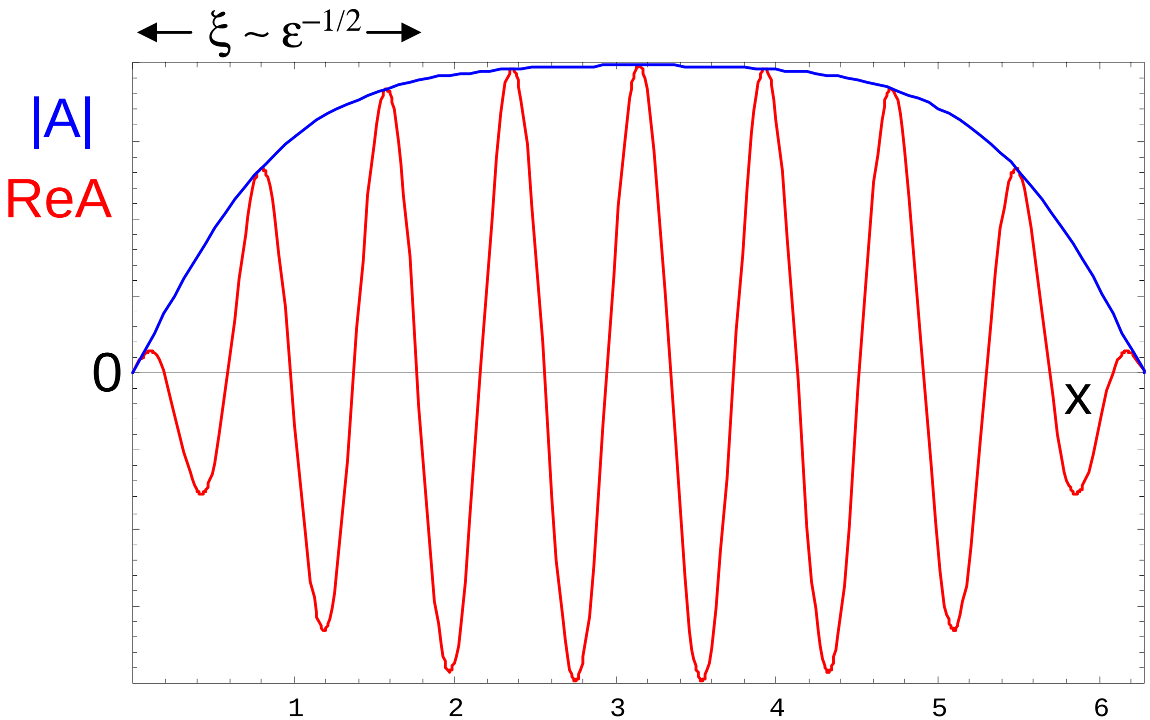


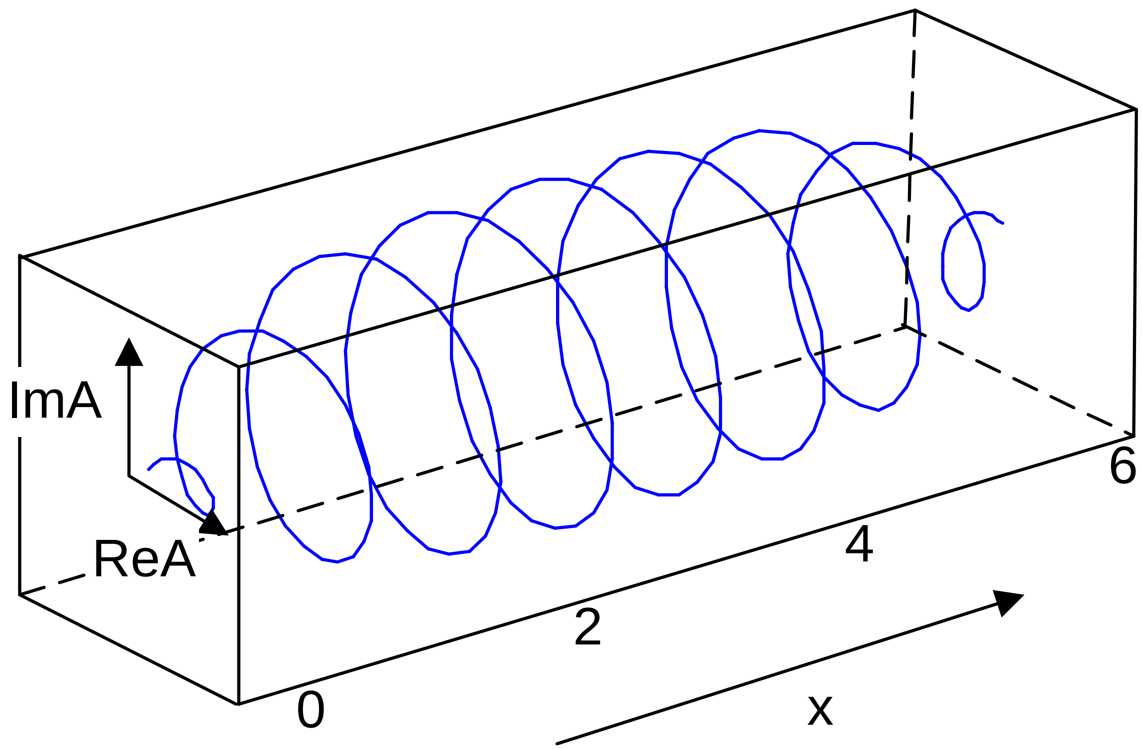
From Morris et al. (1991) and Mao et al. (1996)



Mao et al. (1996)

Two sidewalls




ImA
ReA
0
2
4
6
x

Conclusions

In today's lectures I introduced some of the basic ideas of pattern formation:

- linear instability at nonzero wave number;
- nonlinear saturation;
- stability balloons.

I then introduced the amplitude equation which is the simplest theoretical approach that captures the key effects in pattern formation (growth, saturation, and dispersion).

I focussed on the equation in one dimension, and on a phenomenological derivation. You can find more technical aspects in the supplementary notes.

Next lecture: the role of continuous symmetries — rotation and translation