Lecture 8 Supplementary Notes: Method of Multiple Scales

Michael Cross, 2005

These supplementary notes describe the details of the multiple scales perturbation method. This method is useful in a variety of situations for extracting the slow time dependence of patterns or other systems. The slowness of the time variation may derive from the proximity to a bifurcation, or because of symmetry for example in the situation of gradual spatial distortions of a pattern.

There are two key tricks of the method that may be unfamiliar. The first is the idea of introducing scaled space and time coordinates to capture the slow modulation of the pattern, and treating these as separate variables in addition to the original variables that must be retained to describe the pattern state itself. This is the idea of *multiple scales*. The second is the use of what are known as *solvability conditions* in the formal derivation.

I will formulate the method in terms of deriving the amplitude equation for a rotationally invariant system with a stationary instability to a stripe state. The amplitude equation is an equation derived as an expansion near threshold and in the small size of the magnitude of the spatially periodic deviation from the uniform state. The idea of expanding the nonlinearity in the small amplitude, based on the smallness of the deviation $\varepsilon = (p - p_c)/p_c$ of the control parameter p from the threshold value p_c will not be unexpected. However there are technical differences from more familiar type of perturbation theory often encountered in physics, such as perturbation theory in quantum mechanics on adding a small potential to a Schrödinger equation that can be exactly solved. The difference arises because in the present case the expansion is about a solution to the linear approximation *which has arbitrary size* (since the equation is linear) but the perturbation (the nonlinearity) then fixes the magnitude. Thus we are expanding about an solution with free parameters (the magnitude of the linear solution) which must somehow be fixed within the perturbation scheme.

Multiple Scales

The general framework of the type of problem we are addressing is the set of nonlinear pdes

$$\partial_t \mathbf{u}_p = \mathbf{L}\mathbf{u}_p + \mathbf{N}[\mathbf{u}_p], \tag{1}$$

for the vector of perturbation fields $\mathbf{u}_P(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, t) - \mathbf{u}_b(x_\parallel)$ about the solution $\mathbf{u}_b(x_\parallel)$ which is uniform in the extended directions $\mathbf{x}_\perp$. The extended directions are supposed infinite in extent. In Eq. (1) $\mathbf{L}$ is a linear differential-matrix operator that depends on the control parameter p, and $\mathbf{N}$ collects all the terms that are nonlinear in $\mathbf{u}_p$. The linear instability of the uniform state is signalled by a zero eigenvalue of the operator $\mathbf{L}$ for a perturbation at the critical wave vector $\mathbf{q}_c$. We choose our coordinate system with x along $\mathbf{q}_c$and then the onset of instability is defined by

$$\mathbf{L}_0[e^{iq_c x}\bar{\mathbf{u}}_0(x_\parallel)] = 0 \tag{2}$$

with $\mathbf{L}_0$ equal to $\mathbf{L}$ evaluated at the critical value $p = p_c$ and we choose some convenient normalization convention for $\bar{\mathbf{u}}_0(x_\parallel)$.

Near threshold we expect that the solution might saturate at small magnitude, and we look for the behavior of such solutions. Furthermore we expect the solution to "look like" the linear solution, i.e.
to largely have the same spatial structure. Thus we look for solutions

$$\mathbf{u}_p = \varepsilon^{p_1}\mathbf{u}_0 + \varepsilon^{p_2}\mathbf{u}_1 + \cdots, \tag{3}$$

with p_i some increasing set of powers and $\mathbf{u}_0$ a slow space and time modulation of the critical solution[1]

$$\mathbf{u}_0 = A_0(X, Y, T)e^{iq_c x}\bar{\mathbf{u}}_0(x_\parallel) + \text{c.c.}. \tag{4}$$

It is here that we introduce the multiple scales. The physics that A_0 describes slow modulations of the pattern is introduced at the outset through slow space and time variables, which are traditionally written as upper case letters X, Y, T. The variables X, Y, T are scaled versions of x, y, t so that an $O(1)$ change of X, Y, T corresponds to a small change in the physical variables

$$X = \varepsilon^{-p_x}x, \qquad Y = \varepsilon^{-p_y}y, \qquad T = \varepsilon^{-p_t}t, \tag{5}$$

with the powers p_x, p_y, p_t to be chosen appropriately. We use the symbol A_0 in Eq. (4) because the higher order terms in Eq. (3) contain terms analogous to Eq. (4) but with amplitudes A_i, determined by their own amplitude equations. These amplitudes give additional terms in the amplitude that are higher order in $\varepsilon^{1/2}$.

The powers of ε introduced in Eqs. (3-5) can be found in one of two ways. The first is to leave them as unknown powers, proceed with the expansion, and find what values are needed to get the various terms (nonlinearity, spatial derivatives) to balance in the equation, that is to appear in the resulting equation with coefficients that scale with the same power of ε. The second is to use phenomenological arguments to fix the powers at the outset, with of course the consistency of the formal procedure that follows as a check. I will follow the second approach, since it leads to a more transparent development.

Phenomenological arguments suggest the scaling

$$X = \varepsilon^{1/2}x, \tag{6a}$$

$$Y = \varepsilon^{1/4}y, \tag{6b}$$

$$T = \varepsilon t. \tag{6c}$$

The scaling of the space variables is motivated by the $O(\varepsilon^{1/2})$ width of the wave number band near threshold. As we will see when we talk about symmetry, the different scaling in the x and y direction is motivated by the different dependence of the growth rate on wave vector perturbations in the x and y directions: for $\mathbf{q} = q_c\hat{x} + (Q_x, Q_y)$ the change in growth rate is quadratic in Q_x (hence the $\varepsilon^{1/2}$ scaling of X) but fourth order in Q_y (hence the $\varepsilon^{1/4}$ scaling of Y). Furthermore, the amplitude of saturation is expected to be $O(\varepsilon^{1/2})$, and so we write $\mathbf{u}_p$ as an expansion in powers of $\varepsilon^{1/2}$

$$\mathbf{u}_p = \varepsilon^{1/2}\mathbf{u}_0 + \varepsilon^1\mathbf{u}_1 + \cdots, \tag{7}$$

with $\mathbf{u}_0$ as in Eq. (4) introducing the amplitude A_0.

The scheme is now to substitute Eq. (7) into 1, and collect terms at each order in $\varepsilon^{1/2}$. To do this we must learn how to evaluate the derivatives in $\mathbf{L}$ acting on $\mathbf{u}_p$ expressed in terms of the multiple scales. This follows from the general rule of differentiation: if we have a dependent variable $y(x)$, and a function $f(x, y)$ with dependence on x and y, then the derivative of f with respect to x is

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{dy}{dx}\frac{\partial f}{\partial y}. \tag{8}$$

This is the situation we have, with the dependent variable $X(x) = \varepsilon^{-1/2}x$. It follows that a spatial derivative acting on $\mathbf{u}_p$ can be written

$$\left(\frac{\partial \mathbf{u}_p}{\partial x}\right)_y = \left(\frac{\partial \mathbf{u}_p}{\partial x}\right)_{X,y} + \varepsilon^{1/2}\left(\frac{\partial \mathbf{u}_p}{\partial X}\right)_{x,y}, \tag{9}$$

[1] We use the symbol A_0 here, because the higher order terms in Eq. (**??**) contain terms analogous to Eq. (**??**) but with amplitudes A_i, determined by their own amplitude equations. These amplitudes give additional terms in the ampltidue that are higher order in $\varepsilon^{1/2}$.

or in short

$$\partial_x \to \partial_x + \varepsilon^{1/2}\partial_X \tag{10}$$

where on the right hand side the ∂_x will operate on the $e^{\pm iq_c x}$ dependence and the ∂_X will act on the X dependence of the A_i. Extending this scheme we have

$$\partial_y \to \partial_y + \varepsilon^{1/4}\partial_Y. \tag{11}$$

Higher order derivatives are readily evaluated, e.g.

$$\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 \to \partial_x^2 + \partial_z^2 + \varepsilon^{1/2}(2\partial_x\partial_X + \partial_Y^2) + \varepsilon\partial_X^2. \tag{12}$$

The component at order $\varepsilon^{1/2}$ of Eq. (12), which gives $2iq_c\partial_X + \partial_Y^2$ when acting on a term in $e^{iq_c x}$, will become familiar as the representation at a particular order in ε of the rotationally invariant Laplacian. Similarly the time derivative becomes

$$\partial_t \to \varepsilon\partial_T, \tag{13}$$

since there is no fast time dependence in the base state $e^{iq_c x}$.[2]

Solvability Conditions

The equations of motion are now formally expanded in the small parameter ε, and the terms at each order collected and equated to zero. We first rewrite the general equations of motion as

$$\mathbf{L}\mathbf{u}_p = \partial_t \mathbf{u}_p - \mathbf{N}[\mathbf{u}_p], \tag{14}$$

The linear part of the evolution equation is then expanded in ε using Eqs.(9-13) and the proximity of the control parameter to onset $p = p_c(1+\varepsilon)$ to give

$$\mathbf{L} = \mathbf{L}_0 + \varepsilon^{1/2}\mathbf{L}_1 + \cdots. \tag{15}$$

In particular $\mathbf{L}_0$ is the linearization of the equations of motion about the uniform solution $\mathbf{u} = \mathbf{u}_0$ evaluated at $p = p_c$, as in Eq. (2). Note that the L_i for $i > 0$ will typically contain both fast and slow derivatives (e.g. ∂_x and ∂_X).

Equations (3, 4, 9, and 13) are substituted into Eq. (14), and terms at each order in ε are collected. At order $\varepsilon^{(n+1)/2}$ this will generate an equation for the unknown $\mathbf{u}_n$

$$\mathbf{L}_0\mathbf{u}_n = \text{rhs}. \tag{16}$$

The symbol rhs denotes terms evaluated from lower order calculations depending on $\mathbf{u}_m$, and therefore depending on the amplitudes A_m, for $m < n$. Note that since there are only slow time derivatives $\partial_t \to \varepsilon\partial_T$, the time derivative only appears in rhs as slow time derivatives of the lower order amplitudes.

As well as leading to the solution for $\mathbf{u}_n$, equation (16) actually generates *constraints* on the rhs that are the *solvability conditions* that lead to the important equation for the as yet unspecified amplitude A_0, as well as to equations for its higher order companions A_i when the expansion is continued further. The solvability conditions arise because $\mathbf{L}_0$ has a null space, with at least one eigenvector with zero eigenvalue. We know this to be true because the unstable mode has zero growth rate precisely at onset, and this growth rate is the eigenvalue of $\mathbf{L}_0$; this is the content of Eq. (2). The constraints on the rhs arise because Eq. (16) only has finite solutions for $\mathbf{u}_n$ if the expression rhs has no components in this null space. Technically, this can be

[2] If we were to construct the amplitude equation for an oscillatiry instability this would not be true. In that case, we need to include the fast dependendence ∂_t in the linear operator L since this is an essential part of constructing the linear solution.

expressed by the condition that "rhs is orthogonal to the zero-eigenvalue eigenvector of the adjoint operator $\mathbf{L}_0^\dagger$". This conditions is certainly a mouthful, and you may need to remind yourself from a linear algebra text about the mathematics of resolving a vector along basis vectors, the formal definition of the adjoint operator, and why this statement tells us that rhs then has no components along the zero eigenvector of $\mathbf{L}_0$. Even after understanding the formal content of the expression, since $\mathbf{L}_0$ is in general a matrix differential operator, finding the adjoint and its zero modes is often a difficult calculation in practice. The explicit implementation of the solvability condition in the examples below should help clarify the concepts.

With the solvability condition satisfied, we can formally invert Eq. (16) to give

$$\mathbf{u}_n = \mathbf{L}_0^{-1}(\text{rhs}) + [A_n(X, Y, T)e^{iq_c x}\bar{\mathbf{u}}_0(x_\parallel) + \text{c.c.}], \tag{17}$$

where the second term is the complementary function for the operator $\mathbf{L}_0$ (i.e. the solution to the homogeneous equation given by setting rhs to zero) and introduces an unknown higher-order amplitude function A_n. The solvability conditions at a higher orders eventually lead to the equation for this new amplitude.

This is actually a familiar scheme to those who know secular perturbation theory. We recognize that we are perturbing about the base solution $\mathbf{u}_0$, which however contains a free complex amplitude, corresponding to the arbitrary magnitude of the solution to the linear problem, and the arbitrary position of the stripes. A naïve perturbation expansion will lead to corrections to this base solution that grow without bound as time increases. Secular perturbation theory eliminates these problem terms by placing constraints on the zeroth order solution via the solvability condition: we need to choose the "right" zeroth order solution $\mathbf{u}_0$ in Eq. (7) so that the "correction terms" expressed by the higher order terms are indeed small.

The actual implementation of the scheme for realistic systems is quite involved, even when the base solution is known analytically. I will demonstrate different aspects of the procedure in two examples. First I present an elementary introduction using the Lorenz equations. These are three coupled nonlinear odes that played an important role in the development of chaos theory. This first example illustrates the approach in a simple context, the technique of introducing the slow time scale T, and also promotes an understanding of solvability conditions in the context of matrix equations. However since these equations are ordinary differential equations, they do not illustrate the introduction of the slow spatial dependence. The second example, on the Swift-Hohenberg equation, illustrates the spatial aspects. A more complete example, involving most of the major issues encountered in a typical calculation, is the derivation of the amplitude equation for the full equations of porous convection provided in separate notes.

Amplitude Equations

Lorenz Model

As a simple illustration of the derivation of amplitude equations by the method of multiple scales, we implement the approach for the Lorenz equations for Rayleigh-Bénard convection. These ordinary differential equations are the *starting point* for the subsequent analysis—you do not need to understand their derivation to benefit from the following discussion.

The Lorenz equations are three coupled equations for the components of the vector $\mathbf{u}(t) = (X(t), Y(t), Z(t))$

$$\dot{X} = -\sigma(X - Y), \tag{18a}$$
$$\dot{Y} = rX - Y - XZ, \tag{18b}$$
$$\dot{Z} = b(XY - Z). \tag{18c}$$

where the dot denotes a time derivative. Briefly, the physical content of the equations is the following. In a simple truncated-mode approximation for convection, X represents the circulation velocity in the convection,

Y the spatially periodic temperature perturbation, and Z the heat transport due to the convection. The control parameter r is the reduced Rayleigh number $r = R/R_c$, σ is the Prandtl number, and b is a numerical constant, $b = 8/3$.

A linear stability analysis of Eq. (18) shows that the simple solution $X = Y = Z = 0$ (the uniform, no-convection state) undergoes a bifurcation at the critical value $r_c = 1$ to a state with nonzero X, Y, Z (the convecting state). We wish to derive an amplitude equation that describes the time evolution to the nonlinear state that develops for r slightly larger than r_c. To develop the perturbation expansion we write the bifurcation parameter $r = r_c(1+\varepsilon) = 1+\varepsilon$, and then ε is the small parameter.

We write Eqs. (18) in the form of Eq. (14)

$$\mathbf{L}\mathbf{u}_p = \frac{d\mathbf{u}_p}{dt} - \mathbf{N}[\mathbf{u}_p], \tag{19}$$

with $\mathbf{u}_p$ the perturbation from the base state $\mathbf{u} = 0$, $\mathbf{L}$ the evolution operator acting on $\mathbf{u}_p$, linearized about $X = Y = Z = 0$,

$$\mathbf{L} = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1+\varepsilon & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}, \tag{20}$$

and $\mathbf{N}$ the nonlinear term

$$\mathbf{N}[\mathbf{u}] = \begin{bmatrix} 0 \\ -XZ \\ bXY \end{bmatrix}. \tag{21}$$

The linear operator $\mathbf{L}$ is expanded in powers of $\varepsilon^{1/2}$

$$\mathbf{L} = \mathbf{L}_0 + \varepsilon^{1/2}\mathbf{L}_1 + \varepsilon\mathbf{L}_2 + \cdots, \tag{22}$$

with

$$\mathbf{L}_0 = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}, \quad \mathbf{L}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{L}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{23}$$

The eigenvalues of the matrix $\mathbf{L}_0$ are 0, $-(\sigma+1)$ and $-b$, and the corresponding eigenvectors are $\mathbf{e}_0 = (1,1,0)$, $\mathbf{e}_1 = (-\sigma,1,0)$, and $\mathbf{e}_2 = (0,0,1)$. The zero eigenvalue corresponds to the onset of the linear instability. Also we expand $\mathbf{u}_p = (X,Y,Z)$ in powers of $\varepsilon^{1/2}$

$$\mathbf{u}_p = \varepsilon^{1/2}\mathbf{u}_0(T) + \varepsilon\mathbf{u}_1(T) + \cdots, \tag{24}$$

introducing the slow time variable $T = \varepsilon t$, since we are looking for solutions corresponding to the growth of the weakly unstable mode to saturation. The term $d\mathbf{u}_p/dt$ first contributes at $O(\varepsilon^{3/2})$

$$\frac{d\mathbf{u}_p}{dt} = \varepsilon^{3/2}\frac{d\mathbf{u}_0}{dT} + \varepsilon^2\frac{d\mathbf{u}_1}{dT} + \cdots. \tag{25}$$

These expressions are to be substituted into Eqs. (19), and then we demand that the equation is true at each order in $\varepsilon^{1/2}$.

At $O(\varepsilon^{1/2})$ Eq. (19) reduces to

$$\mathbf{L}_0\mathbf{u}_0 = 0.$$

This shows us that $\mathbf{u}_0$ is simply some amplitude of the zero-eigenvalue mode

$$\mathbf{u}_0 = A_0(T)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \tag{26}$$

which introduces the amplitude function A_0. In the present case A_0 is the amplitude of a real vector, and can be taken as a real function.

At $O(\varepsilon)$ we get (since $\mathbf{Lu} \to \mathbf{L}_0\mathbf{u}_1 + \mathbf{L}_1\mathbf{u}_0$ and $\mathbf{L}_1$ is zero)

$$\mathbf{L}_0\mathbf{u}_1 = \begin{bmatrix} 0 \\ 0 \\ bA_0^2 \end{bmatrix}, \tag{27}$$

where the term in A_0^2 is the $O(\varepsilon)$ nonlinear term, namely bXY evaluated with the solution $\mathbf{u}_0$. The solution of the type of algebraic equation 27 may cause problems, because the matrix $\mathbf{L}_0$ is singular, i.e. it has an eigenvalue that is zero, so that the inverse $\mathbf{L}_0^{-1}$ may not be formed. In the particular case of Eq. (27), we see by inspection that the right hand side has no component along the zero-eigenvalue eigenvector $\mathbf{e}_0 = (1, 1, 0)$ of $\mathbf{L}_0$. In this case Eq. (27) *can* be solved. We find

$$\mathbf{u}_1 = \begin{bmatrix} 0 \\ 0 \\ -A_0^2 \end{bmatrix} + A_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_1 \\ -A_0^2 \end{bmatrix}, \tag{28}$$

where the second term introduces the next order correction $A_1(T)$ to the amplitude that will be determined at a higher order of the expansion: you can check that Eq. (27) is satisfied by Eq. (28) for any value of A_1.

At $O(\varepsilon^{3/2})$ we find

$$\mathbf{L}_0\mathbf{u}_2 = \begin{bmatrix} \partial_T A_0 \\ -\varepsilon A_0 + A_0^3 + \partial_T A_0 \\ bA_0A_1 \end{bmatrix}, \tag{29}$$

where the right hand side gets contributions from $-\mathbf{L}_2\mathbf{u}_0$, $-\mathbf{N}$, and $d\mathbf{u}_p/dt$. Now to solve for $\mathbf{u}_2$ we must explicitly demand that the right hand side of Eq. (29) has vanishing component along the zero-eigenvalue eigenvector $(1, 1, 0)$. This is equivalent to the statement that the right hand side must be orthogonal to the zero-eigenvalue eigenvector of the adjoint operator $\mathbf{L}_0^\dagger$. For a real matrix, the adjoint is just the transpose, so that

$$\mathbf{L}_0^\dagger = \begin{bmatrix} -\sigma & 1 & 0 \\ \sigma & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}. \tag{30}$$

You can check that $\mathbf{L}_0^\dagger$ does indeed have a zero eigenvalue, for which the corresponding eigenvector is $(1, \sigma, 0)$. The condition that the right hand side of Eq. (29) is orthogonal to this vector yields the amplitude equation for A_0

$$\frac{1+\sigma}{\sigma}\partial_T A_0 = \varepsilon A_0 - A_0^3. \tag{31}$$

Equation (31) describes the slow growth and nonlinear saturation of the amplitude of the unstable mode near threshold. In this way, the solvability condition for the existence of the solution $\mathbf{u}_2$ at $O(\varepsilon^{3/2})$ imposes constraints (in the form of a dynamical equation in the slow time dependence) for the amplitude A_0 introduced in the solution $\mathbf{u}_0$. Similarly, extending the procedure to $O(\varepsilon^2)$ would yield a dynamical equation for the next order correction to the amplitude A_1, as well as introducing a further correction A_2, and so on.

This example of the derivation of an amplitude equation introduces many of the features of the full calculation, although there are some simplifications that may not occur in general. For example in the present case $\mathbf{L}_0$ is real. For a real matrix the adjoint is the transpose. In more general examples the operator $\mathbf{L}_0$ will be a complex, matrix differential operator, and we must expand our notions of vector spaces, adjoint operators, etc., in the usual way to function spaces.

One Dimensional Swift-Hohenberg Equation

The one dimensional Swift-Hohenberg equation

$$\partial_t u(x,t) = ru - (\partial_x^2 + 1)^2 u - u^3. \tag{32}$$

was introduced in the lectures as a simple mathematical model displaying the phenomenon of pattern formation. The model has a stationary instability from the uniform state $u = 0$ at the critical value of the control parameter $r = r_c = 0$. The critical wave vector is $q_c = 1$, so that the onset mode is $e^{\pm ix}$. I now show how to derive the lowest order amplitude equation for this model. The new feature that goes beyond the previous example is the spatial dependence in the evolution equation, and in the resulting amplitude equation.

The evolution equation (32) can be written in the general form, Eq. (14), with the linear operator L given by

$$L = (r-1) - 2\partial_x^2 - \partial_x^4, \tag{33}$$

and the nonlinear operator N by

$$N[u_p] = -u_p^3. \tag{34}$$

Following the general procedure outlined at the beginning of these notes, we introduce the small parameter ε as the distance of the control parameter r from its critical value (here $r_c = 0$), $r = \varepsilon$, and then expand the field $u(x,t)$ and the evolution equation in powers of $\varepsilon^{1/2}$. We expand $u(x,t)$ as

$$u = \varepsilon^{1/2} u_0 + \varepsilon u_1 + \text{h.o.t.} \tag{35}$$

with u_0 given as some slowly varying amplitude $A_0(X,T)$ of the critical mode

$$u_0 = A_0(X,T)e^{ix} + \text{c.c.} \tag{36}$$

Here X and T are the slow space and time scales, $X = \varepsilon^{1/2}x$, $T = \varepsilon t$. In the present case, unlike the previous example, the onset mode is complex, and so the amplitude is a complex function. The phase of the complex amplitude is important because it connects to the translational symmetry of the system

In the expansion of the linear operator L in powers of ε, we introduce the multiple scales by the substitution $\partial_x \to \partial_x + \varepsilon^{1/2}\partial_X$. This leads to the replacements

$$\partial_x^2 \to \partial_x^2 + 2\varepsilon^{1/2}\partial_x\partial_X + \varepsilon\partial_X^2, \tag{37a}$$
$$\partial_x^4 \to \partial_x^4 + 4\varepsilon^{1/2}\partial_x^3\partial_X + 6\varepsilon\partial_x^2\partial_X^2 + \cdots. \tag{37b}$$

Thus we find at successive orders in $\varepsilon^{1/2}$

$$L_0 = -1 - 2\partial_x^2 - \partial_x^4, \tag{38a}$$
$$L_1 = -4(\partial_x^2 + 1)\partial_X^2, \tag{38b}$$
$$L_2 = 1 - 2\partial_X^2 - 6\partial_x^2\partial_X^2. \tag{38c}$$

The nonlinear term, and the time derivative $\partial_t u \to \varepsilon\partial_T u$, contribute at $O(\varepsilon^{3/2})$ and higher.

Now collect terms at each order in $\varepsilon^{1/2}$ in the expansion of Eq. (32). At $O(\varepsilon^{1/2})$ we find

$$L_0 u_0 = 0, \tag{39}$$

which is automatically satisfied by the expression Eq. (36). At $O(\varepsilon)$ we have

$$L_0 u_1 = -L_1 u_0 = 0, \tag{40}$$

since the operator $(\partial_x^2 + 1)$ in L_1 gives zero when acting on $e^{\pm ix}$. Thus we simply have

$$u_1 = A_1(X,T)e^{ix} + \text{c.c.}, \tag{41}$$

which introduces the next order correction A_1 to the amplitude. At $O(\varepsilon^{3/2})$, after some effort, we find the equation

$$L_0 u_2 = \left[-(1+4\partial_X^2)A_0 + \partial_T A_0 + 3\left|A_0^2\right| A_0\right] e^{ix} + A_0^3 e^{3ix} + \text{c.c.}. \tag{42}$$

The amplitude equation for A_0 arises as the solvability condition for this equation. The solvability condition arises because the functions $e^{\pm ix}$ satisfy the homogeneous equation

$$L_0 e^{\pm ix} = 0, \tag{43}$$

(i.e. they are zero-eigenvalue eigenvectors for L_0). Thus the coefficient of the $e^{\pm ix}$ dependence on the right hand side of Eq. (42) must be set to zero[3]. This yields the amplitude equation

$$\partial_T A_0 = 1 + 4\partial_X^2 A_0 - 3\left|A_0^2\right| A_0. \tag{44}$$

Returning to the unscaled variables, and writing at lowest order $A = \varepsilon^{1/2} A_0$,yields

$$\partial_t A_0 = \varepsilon A_0 + 4\partial_x^2 A_0 - 3\left|A_0^2\right| A_0, \tag{45}$$

which has the form of the general amplitude equation for a stationary instability to stripes, reduced to one spatial dimension, with values of the parameters

$$\tau_0 = 1, \qquad \xi_0 = 2, \qquad g_0 = 3. \tag{46a}$$

We can easily verify that after the solvability constraint is satisfied, Eq. (42) can indeed be solved. The terms remaining in Eq. (42) are

$$L_0 u_2 = A_0^3 e^{3ix} + \text{c.c.}, \tag{47}$$

which can be solved by inspection to give

$$u_2 = -\frac{1}{64} A_0^3 e^{3ix} + A_1 e^{ix} + \text{c.c.}, \tag{48}$$

with A_1 not determined at this order. Equation (48) can be used to extend the expansion to higher order.

Other Applications of the Solvability Condition

The solvability condition also arises in other situations not arising from a multiple scales expansion. The key ingredient that leads to solvability conditions is the need to invert a linear operator with a zero eigenvalue. In a perturbation context, a zero eigenvalue often arises from a symmetry. For example in a translationally invariant system, the spatial derivative of a stationary localized solution $\mathbf{u}_0(\mathbf{x})$ to

$$\partial_t \mathbf{u} = \mathbf{O}\mathbf{u}(\mathbf{x}, t) \tag{49}$$

satisfies

$$\mathbf{L}\nabla\mathbf{u}_0 = \mathbf{0} \tag{50}$$

where $\mathbf{L}$ is the linear operator given by expanding the operator $\mathbf{O}$ about $\mathbf{u}_0$. An example of this is in the calculation of the climb of dislocations, where we are seeking the dynamics through symmetry related translations along the stripes.

[3]This is easily seen to be same as requiring orthogonality to the zero eigenvalue adjoint eigenvector e^{-ix}.